universität wien

DISSERTATION

Titel der Dissertation

Fermions in Noncommutative Emergent Gravity

angestrebter akademischer Grad

Doktorin der Naturwissenschaften (Dr. rer.nat.)

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Wien, am 17. Februar 2010

Space... The final frontier

Abstract

Noncommutative emergent gravity is a novel framework disclosing how gravity is contained naturally in noncommutative gauge theory formulated as a matrix model. It describes a dynamical space-time which itself is a four-dimensional brane embedded in a higher-dimensional space. In noncommutative emergent gravity, the metric is not a fundamental object of the model, rather it is determined by the Poisson structure and by the induced metric of the embedding.

In this work the coupling of fermions to these matrix models is studied from the point of view of noncommutative emergent gravity. The matrix Dirac operator as given by the IKKT matrix model defines an appropriate coupling for fermions to an effective gravitational metric of noncommutative four-dimensional spaces that are embedded into a ten-dimensional ambient space. As it turns out this coupling is non-standard due to a spin connection that vanishes in the preferred but unobservable coordinates defined by the model.

The purpose of this work is to study the one-loop effective action of this approach. Standard results of the literature cannot be applied due to this special coupling of the fermions. However, integrating out these fields in a nontrivial geometrical background induces indeed the Einstein-Hilbert action of the effective metric, as well as additional terms which couple the noncommutative structure to the Riemann tensor, and a dilaton-like term. It remains to be understood what the effects of these terms are and whether they can be avoided.

In a second step, the existence of a duality between noncommutative gauge theory and gravity which explains the phenomenon of UV/IR mixing as a gravitational effect is discussed. We show how the gravitational coupling of fermions can be interpreted as a coupling of fermions to gauge fields, which suffers then from UV/IR mixing. This explanation does not render the model finite but it reveals why some UV/IR mixing remains even in supersymmetric models, except in the $\mathcal{N} = 4$ case.

Zusammenfassung

Nichtkommutative emergente Gravitation ist ein neuartiger Ansatz zur Verbindung von Gravitation und nichtkommutativer Eichtheorie. Matrixmodelle der Stringtheorie spielen hierbei eine wesentliche Rolle.

Die vorliegende Arbeit behandelt die mathematisch konsistente Einbindung von Fermionen in dieses theoretische Modell. Durch das IKKT-Modell wird ein Dirac-Operator nahe gelegt, der zu einer geeigneten Kopplung der Fermionen an eine effektive Metrik eines vier-dimensionalen, nichtkommutativen Raumes führt, welcher in einen zehn-dimensionalen Raum eingebettet ist. Allerdingst entspricht diese Kopplung nicht der Standardkopplung eines Fermions an einen gekrümmten Raum. Dies ist zurückzuführen auf eine spezielle Spinkonnexion, welche in den vom Matrixmodell definierten Koordinaten verschwindet.

In dieser Arbeit soll die Ein-Schleifen-Quantisierung der fermionischen Wirkung im Rahmen der *nichtkommutativen emergenten Gravitation* bestimmt werden. Aufgrund der anders gearteten Spinkonnexion können Standardresultate der Literatur nicht verwendet werden. Wir zeigen, dass die Ausintegration der fermionischen Felder in einem nicht-trivialen, geometrischen Hintergrund dennoch die zu erwartende Einstein-Hilbert-Wirkung ergibt. Zusätzlich treten aber auch ein dilatonartiger Term auf, sowie ein Term, der die nichtkommutative Struktur an den Riemannschen Krümmungstensor koppelt und der so die Lorentzsymmetrie bricht. Zu klären bleibt, was diese Terme bewirken und ob sie nicht vermieden werden können.

In einem zweiten Schritt wird der Zusammenhang zwischen nichtkommutativer Eichtheorie und Gravitation im Bezug auf das sogenannte *UV/IR mixing* untersucht. Es wird gezeigt, dass die gravitative Kopplung von Fermionen als eine Kopplung von Fermionen an Eichfelder interpretiert werden kann, welche dann unter diesem Mischungsphänomen von ultravioletten und infraroten Divergenzen leidet. *UV/IR mixing* kann demnach als gravitativer Effekt erklärt, wenn auch nicht beseitigt werden - dazu bräuchte es $\mathcal{N} = 4$ Supersymmetrie. Diese Resultate beantworten demnach die Frage, warum *UV/IR mixing* im Falle von $\mathcal{N} < 4$ supersymmetrischen Eichtheorien nicht verschwindet.

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Chapter 1

Introduction

Quantum field theory and general relativity provide the basis for our present understanding of fundamental physics. The latter is a theory of space and time, revealing gravity to be the curvature of a space-time entity. The former describes the three remaining basic forces, the strong, the weak, and the electromagnetic force and their interactions with matter particles[1]. These two theories are independent and on their own consistent frameworks and both have been tested experimentally to a high degree [4, 5]. There is no direct evidence that one of these theories is wrong, at least in the regimes that are within our reach[2]. However, areas in physics in which both quantum and gravitational effects become relevant have not been accessible experimentally so far and it is not expected that they will be in the near future.

But even without direct experimental motive it is conjectured that these two theories should not stand side by side complementing each other but rather that ultimately, one should be able to find a consistent unification of quantum theory and general relativity. This mathematical framework should then stand as a single theory describing nature from the smallest to the largest scales and it is usually called a *quantum theory of gravity*. During the last 70 years tremendous effort was put into a formulation of such a theory and the issue has been attacked from various sides. Next to the most prominent candidates which are string theory and loop quantum gravity there are a number of other approaches. Examples include quantum cosmology, twistor models, supergravity, and notably *noncommutative field theory* which is the approach chosen for this work. However, we have not succeeded so far. A consistent formulation of quantum gravity is not known today.

Why is the construction of quantum gravity so important that generations of physicists have striven and are still striving for it apart from the commonly accepted belief that a theory of everything must exist? On the one hand general relativity predicts its

[1]In principle one can formulate many different quantum field theories. The theory realized in nature is a gauge theory with gauge group $SU(3) \times SU(2)_L \times U(1)_Y$. It is called the Standard Model of particle physics, see [1, 2, 3] for an introduction.

[2]For the Standard Model the next critical test - the prediction of the Higgs boson - will be decided soon when sufficient amount of data of the *Large Hadron Collider* will be available.

own breakdown since under very general conditions, singularities in spacetime where curvature becomes infinite cannot be avoided [6]. Black holes or the big bang are the most popular examples of such a breakdown. Infinities also occur in quantum field theories, where the concept of a point particle leads to divergences in the ultraviolet (UV) regime of the theory. However, in physically sensible models these UV divergences can be treated. This goes under the name of renormalization.

In general, one can say that the uncertainty principle together with general relativity lead to the conclusion that the classical concept of spacetime loses its meaning in the small. When measuring a spacetime coordinate with great accuracy a, there is an uncertainty in the momentum of the order $1/a$. That is to say measuring small distances requires high energies which will curve locally the region of spacetime you want to measure. When the gravitational field becomes so strong as to prevent any signal from escaping that region the operational meaning of this localization gets lost. The process of measuring a spacetime coordinate to infinite accuracy is thus as a matter of principle not possible[3]. It is expected that the conventional concepts of space and time will no longer hold at the Planck scale[4] Λ_{P} and instead some kind of quantum structure of space-time should take over in this regime, introducing naturally a smallest length scale and hence avoiding the problem of singularities of spacetime and smearing out point particles.

What could be the nature of this quantum spacetime? In a seminal work of Doplicher, Fredenhagen, and Roberts [7, 8] the authors explore the limitations of localization measurements which are due to the possible creation of black holes. They find uncertainty relations for the spacetime coordinates x_μ,

$$\begin{aligned} \Delta x_0\left(\Delta x_1+\Delta x_2+\Delta x_3\right) &\gtrsim \Lambda_{\mathrm{P}}^2, \\ \Delta x_1\Delta x_2+\Delta x_2\Delta x_3+\Delta x_3\Delta x_1 &\gtrsim \Lambda_{\mathrm{P}}^2, \end{aligned} \tag{1.1}$$

which are implied by those limitations but do not necessarily imply them. The algebraic structure which in turn implies these uncertainty relations is a *noncommutative algebra* for spacetime coordinates. The basic idea is that the classical spacetime $\mathbb{R}^4$ is replaced by a space $\mathbb{R}^4_\theta$ where the coordinate functions x_μ are promoted to hermitian operators which satisfy Heisenberg-like commutation relations,

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu}, \tag{1.2}$$

where $\theta_{\mu\nu}$ is a real antisymmetric matrix of dimension $(\text{length})^2$. The situation is quite analogous to quantum mechanics. The dimensionful noncommutative parameters $\theta_{\mu\nu}$

[3]This is an argument that goes back to John Archibald Wheeler.

[4]General relativity contains two fundamental parameters, the speed of light c and the gravitational constant G. It is impossible to construct a quantity with the unit of a length out of c and G. Thus general relativity does not distinguish a certain length scale. Taking into account the Planck constant h from quantum mechanics one does obtain a length, the Planck length $\Lambda_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} \sim 1.62 \times 10^{-33}$ cm. This is the regime where quantum gravity should apply.

play the rôle of the reduced Planck constant $\hbar$. The concept of points in spacetime is eliminated, they are replaced by cells whose size is given by the length scale of noncommutativity Λ_{NC} which is of the order $O(\sqrt{\theta})$. The Planck length Λ_{P} is the lower bound for this scale. One can then study so-called *noncommutative quantum field theories* which incorporate quantum fluctuations of spacetime coordinates naturally, see [9, 10], and [11] for a more recent introduction.

One of the main difficulties when formulating a quantum theory of gravity is the different notion of space and time in quantum mechanics, quantum field theory and general relativity. In quantum mechanics we have position and momentum operators that do not commute and time is completely distinguished being an external parameter that is not influenced by any physical event. Time is given so to speak, it is not dynamic. In quantum field theories we still have a fixed Minkowski background and we are dealing with operator-valued fields that are functions of space and time. The main lesson of general relativity on the other hand was that there is no fixed spacetime. Space and time and their geometry are dynamic. Hence one of the main purposes of a quantum theory of gravity should be to clarify what space and time really are in a way that is compatible with both quantum theory and general relativity.

Not only on small scales, even on cosmological scales we find puzzles. With the discovery of cosmic acceleration in 1998 the cosmological constant Λ that appears in the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu} \tag{1.3}$$

came to new fame. A cosmological constant term has the same effect as an energy density of the vacuum. If the energy density is positive, the associated negative pressure will drive an accelerated expansion of space which seems to be observed [12, 13]. The origin of this energy density is not known, hence it is usually called "dark energy". The value of the vacuum energy density can be estimated within quantum field theory where quantum fluctuations of fields up to 300 GeV, which is roughly the highest energy at which current theories have been tested, gives a vacuum energy density of order $(300\mathrm{GeV})^4 \sim 10^{27}$ g/cm^3 [14]. The value of the measured vacuum energy density is about 10^{-29}g/cm^3, i.e. it is smaller by a factor of order 10^{56}. This huge discrepancy between the predicted and the measured value of the vacuum energy density is the so called *cosmological constant problem.* It is considered to be one of the most urging questions in theoretical physics today and a quantum theory of gravity should include a solution for it. For an interpretation of this small energy density, the cosmological constant is near at hand. However, there is by no means evidence that this is the true source of it. The term "dark energy" for this energy density might thus be more appropriate. Of course physics beyond the Standard Model such as supersymmetry could lead to additional contributions to the vacuum energy that might cancel the currently known contributions, but this cancellation would have to be precise to about 56 decimal places. So far we do not know of any mechanism providing such a remarkable cancellation. We will come back to the cosmological constant problem in Sect. 3.5.2

and 3.5.3.

There is many more issues that quantum gravity ought to answer, see [15] for a possible list. However, they are not addressed in this work.

The main objectives of this work. This work is formulated in the framework of "noncommutative emergent gravity" which is a novel framework that discloses gravity in noncommutative gauge theory which is in turn described by a matrix model. The purpose of this work is to study the coupling of fermions in this framework.

This work is organized as follows. In Chapter 2 we give a brief introduction to noncommutative quantum field theory. We discuss the concept of star-products and how one uses them to formulate a field theory. Noncommutative scalar and gauge theories are discussed and they are regarded with respect to their renormalizability properties and the problem of UV/IR mixing.

In Chapter 3 we present the setting within this work was carried out. This framework is called "noncommutative emergent gravity" and it was developed by Harold Steinacker [16] in 2007. We formulate the fundamental ideas which are based on a certain matrix model. It is shown how gravity arises in noncommutative gauge theories which are in turn related to matrix models. In particular, we illustrate how scalar fields and non-Abelian gauge theories are coupled to gravity in this model. The idea of emergent gravity which is crucial for this framework is discussed and a new interpretation for UV/IR mixing in this context is given. We close this part with a discussion of physical solutions and whether this model is an appropriate (toy-?) model for quantum gravity and how it is related to the IKKT model and $\mathcal{N} = 4$ supersymmetry.

This work is devoted to the question how fermions couple to the above mentioned matrix model. The issue is studied in Chapter 4 which is the central part of this work. We will see that the coupling is somewhat unusual since the spin connection vanishes in the coordinate system associated naturally to the model. The crucial question will be whether this model nevertheless induces the correct Einstein-Hilbert action at the one-loop level of perturbation theory. Standard results of the literature cannot be used, instead the induced action has to be evaluated directly. Chapter 4 is devoted to this evaluation.

Chapter 5 deals with the relation between noncommutative $U(1)$ gauge theory and gravity. We will show explicitly for fermions how these two interpretations are just two sides of the same coin. This will give the infamous UV/IR mixing a new interpretation in terms of gravity.

We will finally summarize our results in Chapter 6. An outlook on open issues in the framework of noncommutative emergent gravity will conclude this work.

Chapter 2

Noncommutative quantum field theory

A short introduction to noncommutative quantum field theory is given. We begin with a discussion of the concept of star-products and their relevance in field theories. Major problems such as UV/IR mixing as well as possible solutions are presented. Noncommutative gauge theories and their connections to matrix models close this section. We follow the reviews [9, 11].

2.1 Deforming quantum field theories

To begin with, let us describe how quantum field theories on noncommutative spaces $\mathbb{R}^4_\theta$ are formulated. In a noncommutative world, spacetime coordinates are promoted to operators which fulfill

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}. \tag{2.1}$$

However, instead of working with these operators one can use functions of ordinary spacetime coordinates where the product of the functions is instead modified. Noncommutative quantum field theory (NC QFT) can thus be considered as a deformation of ordinary field theory, see e.g. [9, 10]. Noncommutativity is realized by replacing the usual pointwise product of a pair of fields ϕ and ψ by the so-called "star-product"

$$\phi(x)\psi(x) \to \phi(x) \star \psi(x) = \phi(x)\psi(x) + O(\theta, \partial\phi, \partial\psi). \tag{2.2}$$

The star-product is associative and noncommutative. For $\theta = 0$ the usual product is recovered. In momentum space it becomes

$$\widetilde{\phi}(k)\widetilde{\psi}(q) \to \widetilde{\phi}(k)\widetilde{\psi}(q)e^{\frac{\mathrm{i}}{2}k_\mu\theta^{\mu\nu}q_\nu}, \tag{2.3}$$

where $\widetilde{\phi}$ denotes the Fourier transform of the field ϕ.

A particular example of a star-product is the *Groenewold-Moyal star-product* [17, 18]

$$\begin{aligned}\phi(x) \star \psi(x) &= \phi(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial_\mu}\, \bar{\theta}^{\mu\nu}\, \overrightarrow{\partial_\nu}\right) \psi(x) \\ &= \phi(x)\psi(x) \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}}{2}\right)^n \frac{1}{n!} \bar{\theta}^{\mu_1\nu_1} \dots \bar{\theta}^{\mu_n\nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} \phi(x) \partial_{\nu_1} \dots \partial_{\nu_n} \psi(x).\end{aligned} \tag{2.4}$$

It applies to a constant[1] but possibly degenerate deformation, i.e. $\bar{\theta}$ is not x-dependent and $\bar{\theta}^{-1}$ might not exist. The product Eq. (2.4) gives a deformation of the algebra of functions on $\mathbb{R}^D$ that is unique up to redefinitions of ϕ and ψ that are local order by order in $\bar{\theta}$. As one can see from Eq. (2.4) the star-product contains infinitely-many bi-derivative terms. This makes it a non-local operation which becomes obvious by considering the following integral representation of the star-product:

$$\phi(x) \star \psi(x) = \int\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\mathrm{d}^D k'}{(2\pi)^D} \widetilde{\phi}(k) \widetilde{\psi}(k' - k) e^{-\frac{\mathrm{i}}{2}\bar{\theta}^{\mu\nu} k_\mu k'_\nu} e^{\mathrm{i}k'_\rho x^\rho}. \tag{2.5}$$

To leading order the Groenewold-Moyal star-product coincides with the Poisson bracket of functions with respect to the symplectic form $\bar{\theta}$,

$$\begin{aligned}\phi(x) \star \psi(x) &= \phi(x)\psi(x) + \frac{\mathrm{i}}{2} \bar{\theta}^{\mu\nu} \partial_\mu \phi \partial_\nu \psi + O(\bar{\theta}^2) \\ &= \phi(x)\psi(x) + \frac{\mathrm{i}}{2} \{\phi(x), \psi(x)\} + O(\bar{\theta}^2).\end{aligned} \tag{2.6}$$

The commutator with coordinates x^μ generates derivatives as

$$[x^\mu \stackrel{\star}{,} \phi(x)] := x^\mu \star \phi(x) - \phi(x) \star x^\mu = \mathrm{i}\bar{\theta}^{\mu\nu} \partial_\nu \phi(x), \tag{2.7}$$

and the coordinates fulfill indeed

$$[x^\mu \stackrel{\star}{,} x^\nu] = \mathrm{i}\bar{\theta}^{\mu\nu}. \tag{2.8}$$

Any conventional commutative field theory can now be turned into a noncommutative field theory simply by replacing all products of fields in the action with star-products. However, this process will only affect the interaction terms due to the important property

$$\int \mathrm{d}^D x\, \phi(x) \star \psi(x) = \int \mathrm{d}^D x\, \phi(x)\psi(x), \tag{2.9}$$

which can be shown for Schwartz functions over $\mathbb{R}^D$ using partial integration. Consider for example ϕ^4 scalar field theory in four dimensions. The deformed version writes as

$$S[\phi] = \int \mathrm{d}^4 x \left(\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4}\phi(x) \star \phi(x) \star \phi(x) \star \phi(x)\right). \tag{2.10}$$

[1]Throughout this work constant deformations are denoted as $\bar{\theta}$.

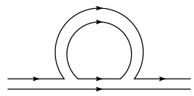

Figure 2.1: One-loop planar ribbon graph contributing to the two-point function in $\phi^{\star 4}$ theory.

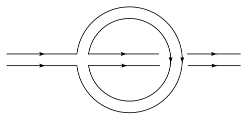

Figure 2.2: One-loop non-planar ribbon graph contributing to the two-point function in $\phi^{\star 4}$ theory. In that case the lines of this ribbon graph do cross.

2.2 UV/IR mixing

We remain with our example of $\phi^{\star 4}$ scalar theory. One can study the perturbation series of such a deformed field theory. The corresponding Feynman rules for this theory were derived by Filk [19] for Euclidean spacetime. In contrast to ordinary field theory where the vertex is given by λ, for NC QFT phase factors appear in the interaction vertex which in momentum space is

$$V(k_1, k_2, k_3, k_4) = \lambda \prod_{a<b} e^{-\frac{\mathrm{i}}{2} k_a \wedge k_b}, \tag{2.11}$$

where

$$k_a \wedge k_b = k_{a\mu} \theta^{\mu\nu} k_{b\nu} = -k_b \wedge k_a. \tag{2.12}$$

Moreover in the noncommutative case, the vertex Eq. (2.11) depends on the (cyclic) ordering of the momenta. In order to account for that feature one doubles the lines of a Feynman diagram making the diagram into a *ribbon graph*. One has to distinguish between *planar* and *non-planar* graphs. Planar graphs can be drawn without crossing lines, as an example see Fig. 2.1. They correspond to graphs of commutative field theory times possible phase factors that depend only on the momenta of the external legs. There are, however, also graphs where the lines do cross as illustrated in the example of Fig. 2.2. These non-planar diagrams contain phase factors given in Eq. (2.11) which depend on the momenta of the internal lines. These phase factors lead to rapid oscillations and thus to a damping of the high-energy behavior of the graph which corresponds to the UV sector of the theory.

Originally, one hoped that noncommutative quantum field theories provide a natural ultraviolet cut-off. Unfortunately, this turned out not to be the case for two reasons. Firstly, since the planar graphs coincide more or less with graphs from commutative field theory, they contain the same UV divergences as the conventional theory. Secondly and even more unfortunately, a new type of divergence appears that mixes divergences for high and low momenta. It can be shown [9] that the effective ultraviolet cutoff for non-planar graphs is of the form

$$\Lambda_{\text{eff}} = \frac{1}{\theta \cdot p}, \tag{2.13}$$

where p denotes the external momentum. One can see that the ultraviolet divergence is restored in either the commutative limit $\theta \to 0$ or in the *infrared limit* $p \to 0$. This means that the effective cutoff Λ_{eff} becomes ineffective for vanishing momenta. This phenomenon goes under the name of *UV/IR mixing* [20]. If one-loop non-planar diagrams are inserted into higher order graphs, this mixing effect will spoil renormalization [21]. This was a severe problem for noncommutative quantum field theories for many years. In recent years progress was achieved, see below. However, the problem has not yet been resolved satisfactorily.

The origin of this obscurity can be found in the non-locality of the theory [11]. If two fields ϕ and ψ are both supported in a small region $\Delta \ll \sqrt{\theta}$, then their star-product $\phi \star \psi$ is non-zero in a large region of size θ/Δ. For a noncommutative field theory this means that very small momenta instantaneously spread out very far upon interaction through the star-product. In other words, high energy processes can have dramatic long-distance effects.

The Grosse-Wulkenhaar model. In order to solve the UV/IR mixing problem one was hoping to find a covariant version of the field theory such that the ultraviolet and the infrared regimes are indistinguishable [22]. In 2003 major progress was achieved when Harald Grosse and Raimer Wulkenhaar presented a model that provided the desired duality which is called the *Langmann-Szabo duality.* The *Grosse-Wulkenhaar model* [23] is a real $\phi^{\star 4}$ scalar theory in a background harmonic oscillator potential in Euclidean four-dimensional spacetime. Its action writes as

$$S[\phi] = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{\Omega^2}{2}(\widetilde{x}_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\right], \tag{2.14}$$

where $\widetilde{x}_\mu = \bar{\theta}^{-1}_{\mu\nu}x^\nu$ and Ω is a constant parameter. The kinetic term of the usual noncommutative scalar theory is replaced by

$$\partial_\mu^2 \to \partial_\mu^2 + \frac{\Omega^2}{2}\widetilde{x}_\mu^2. \tag{2.15}$$

The action Eq. (2.14) is covariant with respect to the Langmann-Szabo duality between position space and momentum space: Under the below exchange of position and momentum,

$$p_\mu \leftrightarrow \widetilde{x}_\mu, \qquad \widehat{\phi}(p) \leftrightarrow \pi^2\sqrt{|\det\bar{\theta}|}\phi(x), \tag{2.16}$$

where $\widehat{\phi}(p_a) = \int \mathrm{d}^4x\, e^{(-1)^a p_{a,\mu} x_a^\mu} \phi(x)$ for a being a cyclic label, the action changes as

$$S[\phi; m, \lambda, \Omega] \mapsto \Omega^2\, S\left[\phi; \frac{m}{\Omega}, \frac{\lambda}{\Omega}, \frac{1}{\Omega}\right]. \tag{2.17}$$

One can see that apart from a scaling factor it has the same appearance in position space and in momentum space.

The success of the Grosse-Wulkenhaar model originates from the fact that it is *renormalizable* to all orders in perturbation theory [24, 23, 25]. For $\Omega = 1$ the model is self-dual under the UV/IR duality. This point in parameter space is exceptional because there the beta-functions for both the coupling constant λ and the oscillator parameter Ω vanish to all orders of perturbation theory [26, 27] which implies that the renormalized coupling flows to a finite bare coupling. This is in contrast to commutative quantum electrodynamics, where the coupling flows to infinity at high but finite energy. The singularities of the renormalization flow are known as *Landau ghosts*. They are related to the occurrence of renormalons which spoil Borel summability. Thus the perturbation series becomes nonsensically at high energies. Landau ghosts can also be found in conventional ϕ^4-theory. The only theory that is well defined perturbatively at high energies in the Standard Model is quantum chromodynamics due to the asymptotic freedom of this theory. The Grosse-Wulkenhaar model is free of renormalons and thus the infamous Landau ghost is tamed, perturbation theory remains valid at all scales. Even a non-perturbative construction of the Grosse-Wulkenhaar model seems feasible and progress has been achieved recently [28].

By now other models have been established which were also proven to be renormalizable [29, 30] due to additional nonlocal terms in the action of the form $1/p^2$. They come with the nice feature of translation invariance which is broken in the case of the Grosse-Wulkenhaar model. In principle, one could add any confining potential to the conventional noncommutative scalar field action Eq. (2.10) to give an effective infrared cutoff. However, the harmonic oscillator potential is the only one providing the desired UV/IR duality [11].

Results in Minkowski spacetime. Up to now renormalization proofs have only been achieved for Euclidean spacetime. As it turns out Minkowski spacetime is incomparably more difficult from a technical point of view than Euclidean spacetime. First it should be mentioned that the Filk's rules [19] for noncommutative scalar theory are valid only for Euclidean spacetime. Corresponding rules in Minkowski space have been worked out by various groups [31, 32], most notably since very concise by Denk and Schweda [33]. Working with the inappropriate rules by Filk [19] leads to a violation of unitarity as noted by Gomis and Mehen [34]. This can be repaired by considering the so-called interaction point time ordering in Minkowski space. If time is noncommutative, i.e. $\theta^{0i} \neq 0$, one has to take into account the different possible time orderings at a vertex. Naive application of perturbation theory à la Feynman is thus not straightforward. In short, a vertex has many "time stamps" which do justice

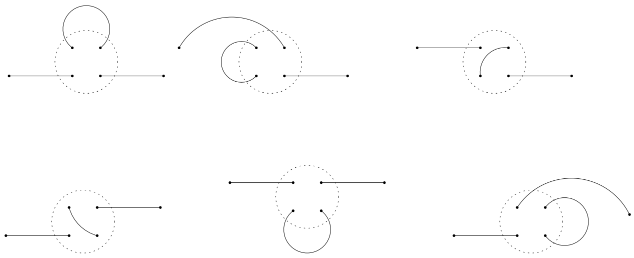

Figure 2.3: Contributions to the tad pole in $\phi^{\star 4}$ theory in Minkowski spacetime for $\theta^{0i} \neq 0$ using the rules of Denk and Schweda [33]. The vertex of the tad pole is denoted by the dashed line. The total amplitude is obtained by summing over all possible time orderings in the vertex.

to the non-locality of the vertex. One has to sum over all possible combinations of these time stamps to obtain the total number of contributions to one graph. Figure 2.3 illustrates an example.

There exist even conjectures that UV/IR mixing might not occur in Minkowski spacetime [35]. However, this needs further investigation. The analytic continuation of UV/IR dual field theories to Minkowski signature has been achieved in [36]. Nevertheless, a renormalization proof seems very far away at the moment.

2.3 Noncommutative gauge theory

Let us now turn to gauge theories on noncommutative spacetime. Due to their prominent rôle in the Standard Model it is of crucial importance to formulate consistent gauge theories also in noncommutative quantum field theories. Following our prescription how to write down the noncommutative version of a conventional field theory we find

$$S = -\frac{1}{4g^2} \int \mathrm{d}^4x \, \mathrm{tr} F_{\mu\nu} \star F^{\mu\nu}, \tag{2.18}$$

where the field strength tensor is defined as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} \left[A_\mu \stackrel{\star}{,} A_\nu \right], \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} \left[A_\mu, A_\nu \right] + O\left(\theta, (\partial A)^2 \right), \end{aligned} \tag{2.19}$$

and g is the gauge coupling. The action Eq. (2.18) is invariant under the following $U(N)$ gauge transformation

$$A_\mu \to U \star A_\mu \star U^{-1} + \mathrm{i} \, U \star \partial_\mu U^{-1} \tag{2.20}$$

with

$$U \star U^\dagger = U^\dagger \star U = 1, \quad U \in U(N). \tag{2.21}$$

A few remarks are in order. Since the star-prodcut is noncommutative, the commutator $[A_\mu \stackrel{\star}{,} A_\nu]$ does not vanish even in the case of Abelian gauge groups. Thus all noncommutative gauge theories are non-Abelian. Moreover the group $SU(N)$ does not close under the star-product and so it is not an appropriate gauge group on $\mathbb{R}^4_\theta$ due to

$$\det(\phi \star \psi) \neq \det(\phi) \star \det(\psi). \tag{2.22}$$

This means that the $U(1)$-sector cannot be decoupled from the $SU(N)$ sector [37] since the $U(1)$ "photon" interacts with the $SU(N)$ gluons. Also, the mixing is limited to the U(1) sector at least at one-loop perturbation theory [37]. We will return to these issues in Sect. 3.2.1 and give a possible explanation.

Noncommutative gauge theory is of exceptional interest because (some version of) gravity is naturally contained in it. The key observation in this context is the fact that spacetime translations of noncommutative gauge fields are equivalent to gauge transformations. This can be seen by the following consideration. An infinitesimal translation $x^\mu \to x^\mu + a^\mu$ acts on functions as

$$f(x^\mu + a^\mu) \sim f(x^\mu) + a^\mu \partial_\mu f(x^\mu). \tag{2.23}$$

For noncommutative coordinates x^μ these are formally inner derivatives because of Eq. (2.7) and

$$\partial_\mu f(x) = \left[-\mathrm{i}\bar{\theta}^{-1}_{\mu\nu} x^\nu, f\right]. \tag{2.24}$$

One obtains then global translations by exponentiating these,

$$\begin{aligned} f(x^\mu + a^\mu) &\sim f(x^\mu) + a^\mu \left[-\mathrm{i}\bar{\theta}^{-1}_{\mu\nu} x^\nu, f\right] \\ &\sim e^{-\mathrm{i}\bar{\theta}^{-1}_{\mu\nu} a^\mu x^\nu} f(x) e^{\mathrm{i}\bar{\theta}^{-1}_{\rho\sigma} a^\rho a^\sigma}. \end{aligned} \tag{2.25}$$

Using the gauge transformation Eq. (2.20) with

$$U(x) = e^{\mathrm{i}\bar{\theta}^{-1}_{\mu\nu} a^\mu x^\nu} \tag{2.26}$$

and the relation Eq. (2.25) we find

$$A_\mu(x) \mapsto A_\mu(x + a) - \bar{\theta}^{-1}_{\mu\nu} a^\nu. \tag{2.27}$$

The constant shift $\bar{\theta}^{-1}_{\mu\nu} a^\nu$ drops out of the field strength tensor. Hence we find that a gauge transformation with $U(x) = e^{\mathrm{i}\bar{\theta}^{-1}_{\mu\nu} a^\mu x^\mu}$ corresponds to a translation $x^\mu \to x^\mu + a^\mu$. Since the action (2.18) is of course gauge invariant, translation symmetry is a gauge symmetry of this action.

In order to obtain general relativity one would need to turn the symmetry under global translations somehow into a local gauge symmetry, and extend the correspondence principle between spacetime and gauge symmetries to the full Poincaré group. This has not been achieved so far and it might not be needed, as we will argue. The precise origin of gravity in noncommutative gauge theory will be discussed in the following chapter.

NC gauge theories, UV/IR mixing and renormalization. The UV/IR mixing problem also occurs in noncommutative gauge theories. The once UV now IR divergences are restricted to the $U(1)$-sector of an $U(N)$ gauge theory at least to one loop [37]. Also here, if statements can be made they are valid only in Euclidean space.

A renormalization proof of noncommutative gauge theories has not been achieved so far. There have been attempts to incorporate the oscillator potential of the Grosse-Wulkenhaar model [38] as well as the $1/p^2$ term [39] to gauge theories and some one-loop calculations have been done [40] showing indeed finiteness of the theories at one-loop. However, it is not clear how to perform a systematic renormalization proof. In the case of gauge theories this might not be necessary anyways, since there the UV/IR mixing has a beautiful interpretation in terms of a gravitational effect, see Sect. 3.3.

2.4 Matrix models

There is a profound relationship between matrix models and noncommutative Yang-Mills theory which is routed in the fact that derivatives ∂_μ can be completely absorbed into the noncommutative gauge fields. This comes as a feature of the noncommutativity of the theory and there is no analog of it for commutative gauge theories. In order to see this we have to introduce the concept of *covariant coordinates* [41]

$$X_\mu = \bar{\theta}^{-1}_{\mu\nu} x^\nu + A_\mu, \tag{2.28}$$

where we assume $\bar{\theta}^{-1}_{\mu\nu}$ to be constant and non-degenerate, and A_μ is a gauge field. Making use of the property Eq. (2.7) the noncommutative field strength tensor of Eq. (2.19) writes then as

$$F_{\mu\nu} = -\mathrm{i}\,[X_\mu \stackrel{\star}{,} X_\nu] + \bar{\theta}^{-1}_{\mu\nu}. \tag{2.29}$$

This means that we can rewrite the noncommutative Yang-Mills theory given by the action Eq. (2.18) in terms of the matrices X_μ only - obtaining the following *matrix model action*

$$S = -\frac{1}{4}\mathrm{Tr}\left(-\mathrm{i}\,[X_\mu, X_\nu] + \bar{\theta}^{-1}_{\mu\nu}\right)^2. \tag{2.30}$$

However, the matrices X_μ can be regarded as abstract objects of either an infinite-dimensional or of a finite-dimensional matrix algebra depending on whether X_μ are finite or infinite matrices. In particular there is no need for any reference to a spacetime dependence. Thus the action Eq. (2.30) can be regarded as background independent, except for $\bar{\theta}^{-1}$ which could be interpreted either as a constant shift or as reference to a Moyal-Weyl background. The action (2.30) is not an unknown model in physics. It is the dimensional reduction of ordinary Yang-Mills theory to a point, which means that the gauge fields do not depend on the spacetime coordinates, with a constant shift $\bar{\theta}^{-1}_{\mu\nu}$. This model is called the twisted reduced model and it is related to the IKKT model for the non-perturbative dynamics of Type IIB superstring theory [42].

Chapter 3

Noncommutative Emergent Gravity

It has been conjectured for a long time that gravity is already contained in noncommutative gauge theory. As we have shown in the last section, noncommutative gauge theory can be rewritten as a matrix model by means of covariant coordinates which describe small fluctuations around spacetime coordinates. From this point of view it is not surprising that quantum fluctuations of spacetime should be related to gravity rather than gauge theory. Rivelles [43] found that noncommutative gauge theory can be regarded as a field theory on a linearized gravitational background which itself depends on the gauge fields. Also Yang has promoted the idea that noncommutative $U(1)$ gauge theory should be viewed as gravity in a series of papers [44, 45, 46, 47]. In order to shed some light on this mechanism which relates noncommutative gauge theory and gravity it is clearly not enough to study quantum field theory on a fixed noncommutative background, since spacetime should become dynamical. Henceforth $\theta^{\mu\nu}$ will be dependent on its position in spacetime, i.e. $\theta^{\mu\nu}(x)$.

In this chapter we formulate the setting within this work was carried out. The framework is called *noncommutative emergent gravity* and it was developed in 2007 by Harold Steinacker in [16]. The ideas were extended in the following in [48, 49, 50, 51, 52]. Recently, cosmological solutions of this model were studied in [53] and mass distributions within our universe were investigated thereafter in [54].

3.1 Fundamental aspects

The central starting point of our approach is the following matrix model

$$S_{\mathrm{YM}} = -(2\pi)^n \,\mathrm{Tr}\left(\frac{1}{4}\left[X^a, X^b\right]\left[X^{a'}, X^{b'}\right]\eta_{aa'}\eta_{bb'}\right). \tag{3.1}$$

The objects X^a are infinite dimensional hermitian matrices or operators acting on a Hilbert space $\mathcal{H}$. The index a runs from 1 to D, where D denotes the number of spacetime dimensions[1]. $\eta_{aa'}$ is an unphysical background metric that fixes

[1] In this work we will consider a certain number of extra dimensions in addition to the usual four dimensions. However, in Sect. 3.3 and Chapter 5 we will study the distinguished case of $D = 4$ which

the signature of the theory, Euclidean $\eta_{ab} = \text{diag}(1, 1, \ldots, 1)$ or Minkowski space $\eta_{ab} = \text{diag}(-1, 1, \ldots, 1)$, respectively.

At the beginning the model contains only matrices which are the dynamical objects in Eq. (3.1). As mentioned before, these matrices can be regarded as completely abstract objects. In particular, there is no spacetime put into the theory, rather this will be a theory of dynamical spacetime itself. This is important for background independence. Here spacetime will not be a fixed background but it will depend on the physical situation. The motivation to study this model comes from the fact that the action given in Eq. (3.1) does occur in string theory. More precisely, this matrix model is related to the IKKT matrix model for the non-perturbative dynamics of Type IIB superstrings [55, 42], see Sect. 3.4.

The matrix model (3.1) under consideration is not identical to the twisted reduced model (2.30) in the previous section. This is because we want to avoid the appearance of $\bar{\theta}^{-1}_{\mu\nu}$ in the covariant coordinates and in the action itself. It seems more natural to start with matrices X^a instead of already inverted quantities. Eq. (3.1) differs from the action Eq. (2.30) only by a constant shift and a topological term of the form

$$[X_a, X_b]\,\bar{\theta}^{ab}, \tag{3.2}$$

which vanishes under the trace.

The intrinsic symmetries of the action Eq. (3.1) are a $U(N)$ gauge symmetry

$$X^a \to U X^a U^\dagger \qquad \text{with} \quad U U^\dagger = U^\dagger U = \mathbb{1}, \tag{3.3}$$

a translational invariance under

$$X^a \to X^a + c^a \qquad \text{for} \quad c^a \in \mathbb{R}^D, \tag{3.4}$$

and an invariance under global $SO(D)$ respectively $SO(1, D-1)$ rotations.

3.1.1 Poisson manifolds

The present approach is based upon the assumption that spacetime carries a Poisson structure

$$\left\{x^a, x^b\right\} = \theta^{ab}(x). \tag{3.5}$$

A manifold that comes with a Poisson structure is called a Poisson manifold $\left(\mathcal{M}, \theta^{ab}(x)\right)$. In our framework spacetime is considered to be the quantization of such a Poisson manifold. The basic idea is that the matrices X^a ought to be seen as the quantization of the coordinate functions x^a of the Poisson manifold $\left(\mathcal{M}, \theta^{ab}\right)$. We take advantage of a seminal work by Kontsevich who has studied the quantization of Poisson manifolds

is important in the context of UV/IR mixing.

intensively [56]. A Poisson structure can be quantized at least locally and there exists a quantization map[2]

$$\mathcal{C}(\mathcal{M}) \to \mathcal{A} \subset L(\mathcal{H}), \tag{3.6}$$

where $\mathcal{C}(\mathcal{M})$ denotes some space of functions on $\mathcal{M}$ and $\mathcal{A}$ denotes the algebra of the functions on $\mathcal{M}$ depending on the quantized coordinates X^a, i.e. $\mathcal{A}$ is the algebra generated by the matrices X^a. For the sake of rigor let us only consider the sub-algebra corresponding to well-behaved functions. We can define a star-product on $\mathcal{C}(\mathcal{M})$, where we assume that this star-product has a meaningful expansion in powers of the Poisson structure θ. Then the commutator of two elements in $\mathcal{A}$ reduces to the Poisson bracket of the classical function on $\mathcal{M}$ to leading order in θ. That is to say one can always choose a star-product such that

$$[f(X), g(X)] = \mathrm{i}\,\{f(x), g(x)\} + O(\theta^2) = \mathrm{i}\theta^{ab}(x)\partial_a(f(x))\partial_b(g(x)) + O(\theta^2). \tag{3.7}$$

This implies the following important relation

$$[X^a, f(X)] = \mathrm{i}\,\{x^a, f(x)\} + O(\theta^2) = \mathrm{i}\theta^{ab}(x)\partial_b f(x) + O(\theta^2). \tag{3.8}$$

It should be stressed that in this work only the semi-classical limit of the model will be considered, i.e. only leading order contributions in θ will be taken into account. At some point corrections to this approximation should be investigated since they could lead to important effects like a possible breaking of Lorentz symmetry. If the energy scale of noncommutativity is sufficiently high it is likely that these effects have not been observable so far. However, these corrections should be worked out in detail if one wants to state quantitative conclusions.

3.1.2 The effective metric $\widetilde{G}^{\mu\nu}$

Before we delve into noncommutative emergent gravity in all its details let us understand how the gravitational coupling on the Poisson manifold $(\mathcal{M}, \theta^{\mu\nu}(x))$ arises. The mechanism was elaborated in a systematic way by Steinacker in [16]. For this purpose we couple a scalar field as a test particle on $\mathcal{M}$ to the matrix model. For the sake of simplicity we will confine ourselves to the case of four dimensions which is stressed by using Greek indices $\mu = 1 \ldots 4$. We will generalize our considerations to extra dimensions in the next sections. A kinetic term for scalar fields is obtained by means of commutators

$$[X^\mu, \Phi] \sim \mathrm{i}\theta^{\mu\nu}\partial_\nu\Phi, \tag{3.9}$$

where $\sim$ denotes the semi-classical limit. Since there is no other way to obtain a kinetic term only fields in the adjoint representation are admissible. The action must hence be of the form

$$S[\Phi] = -(2\pi)^2\,\mathrm{Tr}\,[X^\mu, \Phi]\,[X^\nu, \Phi]\,\eta_{\mu\nu}. \tag{3.10}$$

[2]Weyl quantization for example provides an explicit quantization prescription.

In the semi-classical limit, i.e. to leading order in θ, this action becomes

$$\begin{aligned} S[\Phi] &= -(2\pi)^n \operatorname{Tr}\,[X^\mu,\Phi]\,[X^\nu,\Phi]\,\eta_{\mu\nu} \\ &\sim \int \mathrm{d}^4x\,\rho(x)\,(\theta^{\mu\alpha}\partial_\alpha\Phi(x))\,\big(\theta^{\nu\beta}\partial_\beta\Phi(x)\big)\,\eta_{\mu\nu} \\ &= \int \mathrm{d}^4x\,\rho(x) G^{\mu\nu}\partial_\mu\Phi(x)\partial_\nu\Phi(x), \end{aligned} \tag{3.11}$$

where we have introduced the *effective metric* $G^{\mu\nu}(x)$,

$$G^{\mu\nu}(x) = \theta^{\mu\alpha}(x)\theta^{\nu\beta}(x)\eta_{\alpha\beta}. \tag{3.12}$$

One observes that the scalar matrix model Eq.(3.10) describes in the semi-classical limit a scalar particle coupled to a curved background which is described by an effective metric $G_{\mu\nu}(x)$. Therefore the Poisson manifold acquires naturally a metric structure $(\mathcal{M},\theta^{\mu\nu}(x),G^{\mu\nu}(x))$. However, the effective metric $G^{\mu\nu}(x)$ is not a fundamental object of the theory, rather it is composed by the Poisson structure $\theta^{\mu\nu}$ and the background metric $\eta_{\mu\nu}$ which is flat in the case of four spacetime dimensions. $\theta^{\mu\nu}$ is assumed to be non-degenerate and $\det\theta^{\mu\nu} > 0$. In Eq.(3.11) we also used

$$\begin{aligned} (2\pi)^2 \operatorname{Tr} &\sim \int \mathrm{d}^4x\ (\det\theta^{\mu\nu}(x))^{-1/2}\,, \\ \rho(x) &= (\det\theta^{\mu\nu}(x))^{-1/2} \equiv \Lambda^4_{\mathrm{NC}}(x), \end{aligned} \tag{3.13}$$

where $\rho(x)$ is the symplectic volume. This density factor results from the following consideration [16]: Due to the properties of the trace we have up to boundary terms

$$\operatorname{Tr}\,[f,g] \sim \int \mathrm{d}^4x\,\rho(x)\ \{f,g\} = 0. \tag{3.14}$$

This turns out to fix the density factor $\rho(x)$ up to a constant. The symplectic volume

$$\rho(x)\,\mathrm{d}^4x = \frac{1}{2}\omega^2, \qquad \omega = \mathrm{i}\theta^{-1}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{3.15}$$

satisfies this condition since

$$\begin{aligned} \int \omega^2\,\{f,g\} = \int \omega^2\,X_f[g] &= \int \omega^2\, i_{X_f}\mathrm{d}g \\ &= -\int (i_{X_f}\omega^2)\mathrm{d}g = 2\int (i_{X_f}\omega)\omega\,\mathrm{d}g \\ &= -\int \mathrm{d}f\,\omega\,\mathrm{d}g = 0, \end{aligned} \tag{3.16}$$

which holds up to boundary terms using partial integration. X_f is the Poisson vector field generated by $\{f,.\}$ fulfilling [57]

$$i_{X_f}\omega = -\mathrm{d}f. \tag{3.17}$$

Moreover, $\rho(x)$ can be interpreted as a "local" noncommutative scale $\Lambda_{\rm NC}^4$ since the length scale of noncommutativity $\Lambda_{\rm NC}$ is of the order $O(\sqrt{\theta})$ as argued in Chapter 1. The symplectic density factor $\rho(x)$ is different from the density factor $\sqrt{|G_{\mu\nu}|}$ in general relativity. There $\sqrt{|G_{\mu\nu}|}$ is such that the density factor times the volume element is invariant under spacetime diffeomorphisms,

$$\mathrm{d}^4x\,|G_{\mu\nu}|^{1/2} = \mathrm{d}^4x'|G'_{\mu\nu}(x')|^{1/2}. \tag{3.18}$$

By exploiting the fact that the action Eq.(3.11) is not invariant under Weyl rescaling of $\theta^{\mu\nu}(x)$ and $G^{\mu\nu}(x)$ one can rewrite the action as

$$S[\phi] \sim \int \mathrm{d}^4x\,\sqrt{|\widetilde{G}_{\mu\nu}|}\widetilde{G}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = \int \mathrm{d}^4x\,\widetilde{G}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi, \tag{3.19}$$

where $\widetilde{G}^{\mu\nu}$ is now

$$\widetilde{G}^{\mu\nu}(x) = e^{-\sigma}G^{\mu\nu}(x). \tag{3.20}$$

The scaling factor

$$e^{-\sigma} = (\det\theta^{\mu\nu}(x))^{-1/2} = |\det G_{\mu\nu}(x)|^{1/4} \tag{3.21}$$

is such that $\widetilde{G}^{\mu\nu}$ is unimodular,

$$\det\widetilde{G}^{\mu\nu} = 1. \tag{3.22}$$

In the case of four spacetime dimensions the only way to obtain the standard covariant form of the action is by restricting the class of admissible metrics to unimodular ones. It seems that this class is not rich enough to give physically important solutions such as Schwarzschild [58].

Matrix model coordinates. Let us at this point make the following important remark. Throughout this work we will use so-called "matrix model coordinates" which are preferred coordinates $x^\mu \sim X^\mu$ in the model. Naturally defined by the matrix model, they are such that in the case of four spacetime dimensions the background metric is given by $g_{\mu\nu} = \eta_{\mu\nu}$ respectively $g_{\mu\nu} = \delta_{\mu\nu}$. In the general case of extra dimensions the model allows a $SO(1, D-1)$ respectively $SO(D)$ rotation such that at some given point $p \in \mathcal{M}$ the background metric is again $g_{\mu\nu} = \eta_{\mu\nu}$ respectively $g_{\mu\nu} = \delta_{\mu\nu}$, see Sect. 4.4.

3.1.3 Extra dimensions

The motivation to consider extra dimensions is twofold. On the one hand we have pointed out that there is a close relationship between our framework and string theory, which is known to be formulated consistently only in 10 or 11 dimensional spaces depending on the theory[3]. This issue will be discussed in Sect. 3.4. On the other hand it seems that the four-dimensional framework does not contain enough degrees of freedom to allow for physically important classes of metrics, such as a (perhaps modified) Schwarzschild solution. Furthermore when one wants to derive covariant equations of motions extra dimensions are important. Let us discuss at this point how we mean to implement these additional dimensions into our framework.

Notation. At the beginning we clarify notion. Latin indices $a, b, \dots$ run from 1 to D. From now on Greek indices $\mu, \nu, \dots$ run from 1 to 4. The indices i and j are somewhat exceptional as they run from $1, \dots, D-4$.

We come back to the matrix model action including now one scalar field in four dimensions, i.e. $n = 2$. This action can be regarded as an action of noncommutative gauge theory with one extra dimension if we take the scalar field $\phi = \phi(X^\mu)$ to be our fifth dimension:

$$\begin{aligned} S &= -(2\pi)^2 \operatorname{Tr}\left([X^\mu, X^\rho]\,[X^\nu, X^\sigma]\,\eta_{\mu\nu}\eta_{\rho\sigma} + 2\,[X^\mu, \phi]\,[X^\nu, \phi]\,\eta_{\mu\nu}\right) \\ &= -(2\pi)^2 \operatorname{Tr}\left[X^a, X^{a'}\right]\left[X^b, X^{b'}\right]\eta_{ab}\eta_{a'b'} \qquad \text{with} \quad a, b = 1, \dots, 5. \end{aligned} \tag{3.23}$$

A scalar field can therefore be used to define an *embedding* of a four-dimensional manifold, i.e. a 3-brane, in a higher dimensional space. We generalize this to a $2n-1$ brane in D dimensions, that is a $2n$-dimensional noncommutative space $\mathcal{M}^{2n}_\theta \subset \mathbb{R}^D$ in D dimensions, and study the embedding of this $2n-1$ brane in a higher-dimensional space. By splitting the matrices X^a as

$$X^a = \left(X^\mu, \phi^i\right), \qquad \mu = 1, \dots, 2n, \quad i = 1, \dots, D-2n, \tag{3.24}$$

where the "scalar fields" $\phi^i = \phi^i(X^\mu)$ are assumed to be functions of X^μ which determine the embedding of the $2n$-dimensional submanifold $\mathcal{M}^{2n}$ in $\mathbb{R}^D$. These scalar fields should be regarded as geometrical objects rather than matter fields coupled to the matrix model. Needless to say that one can couple additional matter scalar fields to the model. Expressing the functions $\phi^i(X^\mu)$ in terms of X^μ, we obtain in the semi-classical limit

$$\left[\phi^i(X), f(X)\right] \sim \mathrm{i}\theta^{\mu\nu}(x)(\partial_\mu\phi^i(x))(\partial_\nu f(x)). \tag{3.25}$$

The tangent space of $\mathcal{M}^{2n}_\theta \subset \mathbb{R}^D$ is given by the derivations

$$e^\mu := -\mathrm{i}\,[X^\mu, .] \sim \theta^{\mu\nu}\partial_\nu. \tag{3.26}$$

They define a preferred frame where $\theta^{\mu\nu}(x)$ plays the rôle of a vielbein. However, the distinction between "Lorentz" and "coordinate" indices is not admissible here since neither local Lorentz nor general coordinate transformations are allowed at the outset.

[3] To be precise, noncommutative emergent gravity is related to the IKKT model which is consistent in 10 dimensions. The IKKT model itself is related to M-theory which is formulated in 11 dimensions.

$\mathcal{M}^{2n}$ carries the induced metric

$$g_{\mu\nu}(x) = \frac{\partial x^a}{\partial x^\mu}\frac{\partial x^b}{\partial x^\nu}\eta_{ab} = \eta_{\mu\nu} + (\partial_\mu\phi^i)(\partial_\nu\phi^j)\delta_{ij}. \tag{3.27}$$

In general, the background metric $g_{\mu\nu}(x)$ will be no longer flat as this was the case in four dimensions. This is due to the nontrivial embedding functions $\phi^i(x)$. Note also that $g_{\mu\nu}(x)$ is *not* the metric responsible for the gravitational coupling in the action, since there $g_{\mu\nu}(x)$ will enter only implicitly[4].

The most prominent example for such a spacetime is the 4-dimensional Moyal-Weyl plane, which is a flat manifold with

$$\begin{aligned} [X^\mu, X^\nu] &= \mathrm{i}\bar{\theta}^{\mu\nu}, \qquad \mu,\nu = 0,\dots,3,\\ \phi^i(X) &= 0, \qquad i = 1,\dots,D-4, \end{aligned} \tag{3.28}$$

where $\bar{\theta}$ is constant. However, in general the solutions will fulfill

$$\begin{aligned} [X^\mu, X^\nu] &= \mathrm{i}\theta^{\mu\nu}(X),\\ \phi^i &= \phi^i(X). \end{aligned} \tag{3.29}$$

They describe a dynamical, noncommutative and non-flat four-dimensional manifold with nontrivial embedding in higher dimensions. For reasons that will be discussed in Sect. 3.4, we take

$$D = 10. \tag{3.30}$$

Scalar fields in higher dimensions. We repeat our considerations of Sect. 3.1.2 concerning the arising of the effective metric $\widetilde{G}^{\mu\nu}(x)$ from the matrix model action to elaborate the differences in case of extra dimensions. We split the matrices X^a according to the rule above Eq. (3.24)

$$\begin{aligned} S[\Phi] &= -(2\pi)^n \operatorname{Tr}\left([X^a,\Phi]\left[X^b,\Phi\right]\eta_{ab}\right)\\ &= -(2\pi)^n \operatorname{Tr}\left([X^\mu,\Phi]\left[X^\nu,\Phi\right]\eta_{\mu\nu} + \left[\phi^i,\Phi\right]\left[\phi^j,\Phi\right]\delta_{ij}\right)\\ &\sim (2\pi)^n \operatorname{Tr}\left(\theta^{\mu\alpha}\theta^{\nu\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi)\eta_{\mu\nu} + \theta^{\mu\alpha}\theta^{\nu\beta}(\partial_\mu\phi^i)(\partial_\alpha\Phi)(\partial_\nu\phi^j)(\partial_\beta\Phi)\delta_{ij}\right)\\ &= \int \mathrm{d}^4x\, \rho(x)\, G^{\mu\nu}(x)\partial_\mu\Phi\partial_\nu\Phi. \end{aligned} \tag{3.31}$$

To avoid confusion we stress that now there are *two types* of scalar fields present: The scalar field $\Phi(x)$ which is a matter field and the scalar fields $\phi^i(x)$ which come from the embedding and which are considered to be geometrical objects. We observe once

[4]This is true for scalar fields and gauge fields. In the case of fermions this holds at tree-level. However, the background metric enters the one-loop effective action via the Riemann tensor $R_{\mu\nu\rho\sigma}[g]$ explicitly, see Sect. 4.4.

more that in the semi-classical limit the matter scalar field Φ couples to an effective metric of the form

$$G^{\mu\nu}(x) = \theta^{\mu\alpha}(x)\theta^{\nu\beta}(x)g_{\mu\nu}(x). \tag{3.32}$$

$G^{\mu\nu}(x)$ is now determined by two dynamical objects, the Poisson structure $\theta^{\mu\nu}(x)$ and the induced metric $g_{\mu\nu}(x)$. The density factor $\rho(x)$ is again given by the symplectic volume

$$\rho(x) = |\theta^{-1}_{\mu\nu}(x)|^{1/2} = |G_{\mu\nu}(x)|^{1/4}|g_{\mu\nu}(x)|^{1/4} \equiv \Lambda^4_{\mathrm{NC}}(x). \tag{3.33}$$

Weyl rescaling of the metric $G^{\mu\nu}(x)$ by $e^{-\sigma}$

$$\begin{aligned} G^{\mu\nu}(x) \to \widetilde{G}^{\mu\nu}(x) &= e^{-\sigma}G^{\mu\nu}(x), \\ \widetilde{G}^{\mu\nu}(x) &= e^{-\sigma}\theta^{\mu\alpha}(x)\theta^{\nu\beta}(x)g_{\alpha\beta}(x) \end{aligned} \tag{3.34}$$

results once more in a correct covariant scalar action,

$$S[\Phi] = \int \mathrm{d}^4x\, |\widetilde{G}_{\mu\nu}|^{1/2}\, \widetilde{G}^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi), \tag{3.35}$$

where the scaling factor is now given by

$$e^{-\sigma} = \frac{|G_{\mu\nu}(x)|^{1/4}}{|g_{\mu\nu}(x)|^{1/4}} = \frac{|\theta^{-1}_{\mu\nu}(x)|^{1/2}}{|g_{\mu\nu}(x)|^{1/2}}. \tag{3.36}$$

It has the same form as the four-dimensional action of Eq. (3.10), except that the constant background metric $\eta_{\mu\nu}$ is now replaced by the induced metric $g_{\mu\nu}(x)$. As a consequence we are *not restricted* to the class of unimodular metrics when working with extra dimensions.

Yang-Mills action in higher dimensions. The semi-classical limit of the matrix model action Eq. (3.1) is given by

$$S_{\mathrm{YM}} = -(2\pi)^n\, \mathrm{Tr}[X^a, X^{a'}][X^b, X^{b'}]\eta_{ab}\eta_{a'b'} \sim 4\int \mathrm{d}^{2n}x\, \rho(x)\, \eta(x), \tag{3.37}$$

where we have used

$$\begin{aligned} \left[X^a, x^{a'}\right]\left[X^b, X^{b'}\right]\eta_{ab}\eta_{a'b'} &= \left[X^\mu, X^\rho\right]\left[X^\nu, X^\sigma\right]\eta_{\mu\nu}\eta_{\rho\sigma} \\ &\quad + 2\left[X^\mu, \phi^i\right]\left[X^\nu, \phi^j\right]\eta_{\mu\nu}\delta_{ij} \\ &\quad + \left[\phi^i, \phi^m\right]\left[\phi^j, \phi^n\right]\delta_{ij}\delta_{mn} \\ &\sim -\theta^{\mu\rho}\theta^{\nu\sigma}\eta_{\mu\nu}\eta_{\rho\sigma} - 2\theta^{\mu\rho}(\partial_\rho\phi^i)\theta^{\nu\sigma}(\partial_\sigma\phi^j)\eta_{\mu\nu}\delta_{ij} \\ &\quad - \theta^{\mu\rho}(\partial_\mu\phi^i)(\partial_\rho\phi^m)\theta^{\nu\sigma}(\partial_\nu\phi^j)(\partial_\sigma\phi^n)\delta_{ij}\delta_{mn} \\ &= -\theta^{\mu\rho}\theta^{\nu\sigma}g_{\mu\nu}g_{\rho\sigma} \\ &= -G^{\mu\nu}(x)g_{\mu\nu}(x) \\ &\equiv -4\,\eta(x). \end{aligned} \tag{3.38}$$

This can be written in covariant manner as

$$\begin{aligned}S_{\rm YM} = \int \mathrm{d}^4x\,\rho(x)G^{\mu\nu}g_{\mu\nu} = \int \mathrm{d}^4x\,\rho(x)e^{\sigma}\widetilde{G}^{\mu\nu}g_{\mu\nu} = \int \mathrm{d}^4x\,|\widetilde{G}_{\mu\nu}|^{1/2}\widetilde{G}^{\mu\nu}g_{\mu\nu} \\ = 4\int \mathrm{d}^4x\,|\widetilde{G}_{\mu\nu}|^{1/2}\,\widetilde{\eta}(x),\end{aligned} \tag{3.39}$$

where

$$\widetilde{\eta}(x) = \frac{1}{4}\widetilde{G}^{\mu\nu}g_{\mu\nu}. \tag{3.40}$$

The Yang-Mills matrix action in the semi-classical limit is a purely geometrical object. If $\widetilde{G}^{\mu\nu} = g_{\mu\nu}$ it corresponds to a brane tension.

3.1.4 Equations of motion

In the matrix model Eq. (3.1) a priori there is no geometry, all we have is matrices. The geometry of this model arises dynamically. The matrix model is therefore a theory of spacetime itself, in the sense that the physically realized geometry has to fulfill the equations of motion (e.o.m.) of the theory which on matrix level are given by

$$\left[X^a, \left[X^b, X^{a'}\right]\right]\eta_{aa'} = 0. \tag{3.41}$$

We want to obtain the semi-classical version of this equation.

Equations of motion for $\theta^{-1}_{\mu\nu}(x)$. First we evaluate the equations of motion of the tangential components X^μ from the matrix model Eq. (3.1):

$$\begin{aligned}\left[X^a, \left[X^\mu, X^{a'}\right]\right]\eta_{aa'} &= \left[X^\rho, \left[X^\mu, X^\sigma\right]\right]\eta_{\rho\sigma} + \left[\phi^i, \left[X^\nu, \phi^j\right]\right]\delta_{ij} \\ &\sim -\theta^{\rho\lambda}\left(\partial_\lambda\theta^{\mu\sigma}\right)\eta_{\rho\sigma} - \theta^{\rho\lambda}(\partial_\rho\phi^i)(\partial_\lambda\theta^{\mu\sigma})(\partial_\sigma\phi^j)\delta_{ij} \\ &\quad - \theta^{\nu\eta}(\partial_\nu\phi^i)\theta^{\mu\lambda}(\partial_\eta\partial_\lambda\phi^j)\delta_{ij} \\ &= \theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})g_{\alpha\beta} + \theta^{\rho\alpha}\theta^{\mu\beta}(\partial_\rho g_{\alpha\beta}) \\ &= 0,\end{aligned} \tag{3.42}$$

where we have used

$$\partial_\eta g_{\lambda\nu} = (\partial_\eta\partial_\lambda\phi^i)(\partial_\nu\phi^j)\delta_{ij} + (\partial_\lambda\phi^i)(\partial_\eta\partial_\nu\phi^j)\delta_{ij}. \tag{3.43}$$

We find that the equation of motion for $\theta^{\mu\nu}(x)$ in the semi-classical limit is given by

$$\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})g_{\alpha\beta} + \theta^{\rho\alpha}\theta^{\mu\beta}(\partial_\rho g_{\alpha\beta}) = 0, \tag{3.44}$$

which is equivalent to

$$G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\alpha}) = \theta^{\rho\sigma}(\partial_\rho g_{\sigma\alpha}). \tag{3.45}$$

Note that this form of the "tangential e.o.m." is very useful for our considerations later. However, this equation is valid only in matrix model coordinates. It has been shown by Steinacker [50] that it is possible to write this equation in covariant form as

$$\widetilde{G}^{\rho\sigma}\widetilde{\nabla}_\rho\left(e^\sigma\theta^{-1}_{\sigma\nu}\right) = e^{-\sigma}\widetilde{G}_{\mu\nu}\theta^{\mu\alpha}\partial_\alpha\eta(x). \tag{3.46}$$

Here $\widetilde{\nabla}_\rho$ denotes the Levi-Civita connection with respect to the effective metric $\widetilde{G}^{\mu\nu}$ of Eq. (3.34). The above equation of motion (3.46) determines the dynamics of spacetime itself (however, without matter coupled). In this respect it is the analogue of the Einstein equation of general relativity. The equation looks very different compared to the Einstein equation due to the fact that the metric is not the fundamental object of the model.

Equations of motion for scalar fields. We turn now to the equations of motion for scalar particles starting from the matrix model stated in Eq. (3.31):

$$\begin{aligned}
\left[X^a,\left[X^b,\Phi\right]\right]\eta_{ab} &= \left[X^\mu,\left[X^\nu,\Phi\right]\right]\eta_{\mu\nu} + \left[\phi^i,\left[\phi^j,\Phi\right]\right]\delta_{ij} \\
&\sim -\theta^{\mu\sigma}\partial_\sigma\left(\theta^{\nu\rho}(\partial_\rho\Phi)\right)\eta_{\mu\nu} - \theta^{\rho\sigma}(\partial_\rho\phi^i)\partial_\sigma\left(\theta^{\mu\nu}(\partial_\mu\phi^j)(\partial_\nu\Phi)\right)\delta_{ij} \\
&= -\theta^{\sigma\mu}(\partial_\sigma\theta^{\rho\nu})(\partial_\rho\Phi)\eta_{\mu\nu} - \theta^{\nu\sigma}(\partial_\nu\phi^i)(\partial_\sigma\theta^{\mu\rho})(\partial_\mu\phi^j)(\partial_\rho\Phi)\delta_{ij} \\
&\quad - \theta^{\mu\sigma}\theta^{\nu\rho}(\partial_\rho\partial_\sigma\Phi)\eta_{\mu\nu} - \theta^{\rho\mu}\theta^{\sigma\nu}(\partial_\rho\phi^i)(\partial_\sigma\phi^j)(\partial_\mu\partial_\nu\Phi)\delta_{ij} \\
&\quad - \theta^{\rho\sigma}\theta^{\mu\nu}(\partial_\sigma g_{\rho\mu})(\partial_\nu\Phi) \\
&= -G^{\mu\nu}(\partial_\mu\partial_\nu\Phi) - \theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})g_{\alpha\beta}(\partial_\nu\Phi) \\
&\quad - \theta^{\mu\alpha}\theta^{\nu\beta}(\partial_\mu g_{\alpha\beta})(\partial_\nu\Phi) \\
&= 0.
\end{aligned} \tag{3.47}$$

If we express the contracted Christoffel symbol $\widetilde{G}^{\rho\sigma}\widetilde{\Gamma}^\mu_{\rho\sigma}$ in terms of the Poisson structure and the background metric $g_{\mu\nu}(x)$ we find

$$\begin{aligned}
\widetilde{\Gamma}^\mu &= \widetilde{G}^{\rho\sigma}\widetilde{\Gamma}^\mu_{\rho\sigma} \\
&= -\partial_\rho\widetilde{G}^{\rho\mu} - \frac{1}{2}\widetilde{G}^{\mu\nu}\widetilde{G}^{\rho\sigma}\partial_\nu\widetilde{G}_{\rho\sigma} \\
&= -e^{-\sigma}\theta^{\nu\alpha}\partial_\nu\left(\theta^{\mu\beta}g_{\alpha\beta}(x)\right) \\
&= -e^{-\sigma}\left(\theta^{\nu\alpha}(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} + \theta^{\nu\alpha}\theta^{\mu\beta}(\partial_\nu g_{\alpha\beta})\right).
\end{aligned} \tag{3.48}$$

The equation of motion (3.44) in matrix coordinates is thus equivalent to the non-covariant equation

$$\widetilde{\Gamma}^\mu = 0. \tag{3.49}$$

The semi-classical equation of motion for a scalar is hence given by

$$\begin{aligned}
0 &= G^{\mu\nu}(\partial_\mu\partial_\nu\Phi) + \theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})g_{\alpha\beta}(\partial_\nu\Phi) + \theta^{\mu\alpha}\theta^{\nu\beta}(\partial_\mu g_{\alpha\beta})(\partial_\nu\Phi) \\
&= e^\sigma\widetilde{G}^{\mu\nu}(\partial_\mu\partial_\nu\Phi) - e^\sigma\widetilde{\Gamma}^\mu(\partial_\mu\Phi) \\
&= e^\sigma\left(\widetilde{G}^{\mu\nu}\partial_\mu\partial_\nu - \widetilde{\Gamma}^\mu\partial_\mu\right)\Phi,
\end{aligned} \tag{3.50}$$

which is the Laplace-Beltrami operator acting on Φ. Our final form for the e.o.m. for a scalar field is thus

$$\Delta_{\widetilde{G}}\, \Phi = \left(\widetilde{G}^{\mu\nu}\partial_\mu\partial_\nu - \widetilde{\Gamma}^\mu\partial_\mu\right)\Phi = 0. \tag{3.51}$$

The same argument as above gives the equations of motion for the embedding functions $\phi^i(x)$ in the matrix model Eq. (3.1),

$$\Delta_{\widetilde{G}}\, \phi^i = 0, \tag{3.52}$$

and similarly for $x^\mu \sim X^\mu$,

$$\Delta_{\widetilde{G}}\, x^\mu = 0. \tag{3.53}$$

These equations express the freedom of choosing the separation of $X^a = (X^\mu, \phi^i(X^\mu))$ into coordinates and embedding scalar fields. Moreover we notice that on-shell geometries which fulfill Eq. (3.44) imply *harmonic coordinates*

$$\widetilde{G}^{\mu\nu}\partial_\mu\partial_\nu\phi^i = 0. \tag{3.54}$$

In general relativity this would be interpreted as a gauge condition.

Noether theorem. In [51] it was shown that the equations of motion for the Poisson structure follow from a *matrix Noether theorem*[5]. They are therefore protected from quantum corrections. This result came somewhat surprising and the physical mechanism responsible is not fully understood. To see this, recall that the matrix model Eq. (3.1) is invariant under a translational symmetry

$$X^a \to x^a + c^a \mathbb{1}, \qquad c^a \in \mathbb{R}^D. \tag{3.55}$$

This symmetry goes along with the following conservation law

$$\left[X^a, T^{bc}\right]\eta_{ab} = 0, \tag{3.56}$$

which can be verified using the matrix equations of motion (3.41). In the above equation, T^{ab} corresponds to the matrix "energy-momentum tensor"

$$\begin{aligned} T^{ab} = {} & \left[X^a, X^c\right]\left[X^b, X^d\right]\eta_{cd} + \left[X^b, X^c\right]\left[X^a, X^d\right]\eta_{cd} \\ & - \frac{1}{2}\eta^{ab}\left[X^c, X^d\right]\left[X^{c'}, X^{d'}\right]\eta_{cc'}\eta_{dd'}. \end{aligned} \tag{3.57}$$

In the semi-classical limit it can be written in terms of geometrical expressions

$$\begin{aligned} T^{\mu\nu} &\sim -2G^{\mu\nu}(x) + 2\eta^{\mu\nu}\eta(x), \\ T^{\mu i} &\sim -2G^{\mu\nu}\partial_\nu\phi^i(x), \\ T^{ij} &\sim -2G^{\mu\nu}(\partial_\mu\phi^i)(\partial_\nu\phi^j) + 2\delta^{ij}\eta(x). \end{aligned} \tag{3.58}$$

[5] Also the e.o.m. for the non-Abelian gauge fields follow from a matrix Noether theorem, see Sect. 3.2.2.

We are interesseted in the conservation law Eq. (3.56) at semi-classical level. The tangential law is given by

$$\begin{aligned}0 &= \left[X^\mu, T^{\nu\rho}\right]\eta_{\mu\nu} + \left[\phi^i, T^{j\rho}\right]\delta_{ij} \\ &\sim \mathrm{i}\theta^{\mu\lambda}\partial_\lambda\left(-2G^{\nu\rho} + 2\eta^{\nu\rho}\eta(x)\right)\eta_{\mu\nu} \\ &\quad + \mathrm{i}\theta^{\mu\nu}(\partial_\mu\phi^i)\partial_\nu\left(-2G^{\rho\lambda}(\partial_\lambda\phi^j)\right)\delta_{ij},\end{aligned} \tag{3.59}$$

which can be reduced to

$$\theta^{\mu\lambda}(\partial_\lambda G^{\nu\rho}g_{\mu\nu}) - \theta^{\rho\sigma}\partial_\sigma\eta(x) = 0. \tag{3.60}$$

One can show that this equation coincides precisely with the covariant equation of motion Eq. (3.46). In order to see this we need a couple of relations. First note that

$$\partial_\mu\theta^{\mu\nu} = -\theta^{\mu\nu}\rho^{-1}\partial_\mu\rho \tag{3.61}$$

which follows from

$$\begin{aligned}\partial_\mu(\rho\theta^{\mu\nu}) &= (\partial_\mu\rho)\theta^{\mu\nu} + \rho(\partial_\mu\theta^{\mu\nu}) \\ &= -\frac{1}{2}\sqrt{\det\theta^{-1}}\theta^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\sigma})\theta^{\mu\nu} + \rho(\partial_\mu\theta^{\mu\nu}) \\ &= \frac{1}{2}\rho\left(\theta^{\rho\sigma}(\partial_\sigma\theta^{-1}_{\mu\rho}) + \theta^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\mu})\right)\theta^{\mu\nu} + \rho(\partial_\mu\theta^{\mu\nu}) \\ &= \rho\,\theta_{\mu\alpha}(\partial_\rho\theta^{\rho\alpha})\theta^{\mu\nu} + \rho(\partial_\mu\theta^{\mu\nu}) \\ &= 0,\end{aligned} \tag{3.62}$$

where we have used the Jacobi identity. Next we introduce the antisymmetric matrix

$$\hat{\theta}^{\mu\nu} = G^{\mu\rho}g_{\rho\sigma}\theta^{\sigma\nu} = -G^{\mu\rho}\theta^{-1}_{\rho\sigma}G^{\sigma\nu}. \tag{3.63}$$

With the help of $\hat{\theta}^{\mu\nu}$ we can write the tangential conservation law as

$$\partial_\mu\left(\rho\,\hat{\theta}^{\mu\nu}\right) = \rho\,\theta^{\mu\nu}(\partial_\mu\eta(x)). \tag{3.64}$$

Now consider the covariant derivative of $\hat{\theta}^{\mu\nu}$

$$\widetilde{\nabla}_\mu\hat{\theta}^{\mu\nu} = \partial_\mu\hat{\theta}^{\mu\nu} + \widetilde{\Gamma}^\mu_{\mu\lambda}\hat{\theta}^{\lambda\nu} \tag{3.65}$$

$$= \partial_\mu\hat{\theta}^{\mu\nu} + \frac{1}{2}\hat{\theta}^{\lambda\nu}(\widetilde{G}^{\rho\sigma}\partial_\lambda\widetilde{G}_{\rho\sigma}) \tag{3.66}$$

$$= \hat{\theta}^{\lambda\nu}\frac{1}{\sqrt{|\widetilde{G}|}}\partial_\lambda|\widetilde{G}|^{1/2}. \tag{3.67}$$

Therefore we find

$$\sqrt{|\widetilde{G}|}\widetilde{\nabla}_\mu\hat{\theta}^{\mu\nu} = \partial_\mu(\hat{\theta}^{\mu\nu}\sqrt{|\widetilde{G}|}) \tag{3.68}$$

or

$$\partial_\mu(e^{-\sigma}\sqrt{|\widetilde{G}|}\hat{\theta}^{\mu\nu}) = \partial_\mu(\rho\,\hat{\theta}^{\mu\nu}) = \sqrt{|\widetilde{G}|}\widetilde{\nabla}_\mu\left(e^{-\sigma}\hat{\theta}^{\mu\nu}\right), \tag{3.69}$$

where we remembered $|\widetilde{G}|^{1/2} = \rho\, e^{\sigma}$. At the end we obtain the following form for the tangential conservation law Eq. (3.64)

$$\widetilde{\nabla}_\mu(e^{-\sigma}\hat{\theta}^{\mu\nu}) = e^{-\sigma}\theta^{\mu\nu}\partial_\mu\eta(x). \tag{3.70}$$

Using Eq. (3.63) we finally see that this form is equivalent to the covariant equation of motion (3.46). As a consequence the covariant equation of motions also apply at the quantum level and anomalies are not expected.

Let us complete these considerations by quoting the conservation law for the remaining scalar directions

$$\begin{aligned}
-\mathrm{i}[X^\mu, T^{\nu i}]\eta_{\mu\nu} - \mathrm{i}[\phi^m, T^{ni}]\delta_{mn} &= -2\theta^{\eta\sigma}G^{\mu\rho}g_{\eta\mu}\partial_\sigma\partial_\rho\phi^i - 2\theta^{\eta\sigma}\partial_\sigma(G^{\mu\rho}g_{\eta\mu})\partial_\rho\phi^i \\
&\quad + 2\theta^{\rho\sigma}(\partial_\sigma\eta(x))(\partial_\rho\phi^i) \\
&= -2\theta^{\eta\sigma}g_{\eta\mu}G^{\mu\rho}\partial_\sigma\partial_\rho\phi^i \\
&= 2\hat{\theta}^{\rho\sigma}\partial_\rho\partial_\sigma\phi^i \\
&= 0.
\end{aligned} \tag{3.71}$$

The scalar conservation law is a consequence of the tangential conservation law Eq. (3.60) which we used in the last step. As such it does not give an additional condition. Whenever the tangential conservation law is fulfilled automatically so is the scalar one.

3.2 Geometry and gauge theory

3.2.1 Geometry from $U(1)$ gauge fields

So far we have discussed how (some form of) gravity is contained naturally in matrix models. Remember now that we have already mentioned an important relationship between noncommutative gauge theory and matrix models in Sect. 2.4: One can rewrite noncommutative gauge theory as a matrix model. The action Eq. (3.1) should thus be interpretable as an action for a noncommutative gauge theory. In order to study this relationship we will need once more the concept of covariant coordinates. Intuitively, it is not surprising that noncommutative gauge theories have something to do with gravity since covariant coordinates describe fluctuations of spacetime coordinates with the help of a gauge field. For the sake of simplicity we restrict ourselves in this section to four spacetime dimensions. The background metric $g_{\mu\nu} = \eta_{\mu\nu}$ is then constant.

During the last sections we were dealing with noncommutative spacetimes defined by the relation

$$[X^\mu, X^\nu] = \mathrm{i}\theta^{\mu\nu}(X). \tag{3.72}$$

It was important that the Poisson tensor was x-dependent in order to obtain a dynamical effective metric

$$\widetilde{G}^{\mu\nu}(x) = e^{-\sigma}\theta^{\mu\alpha}(x)\theta^{\nu\beta}(x)\eta_{\alpha\beta}. \tag{3.73}$$

In oder to make contact to nonocmmutative gauge theory we consider now small fluctuations $\mathcal{A}^\mu$ around flat Moyal-Weyl space $\mathbb{R}^4_{\bar{\theta}}$. The generators $\bar{X}^\mu$ of this flat but noncommutative spacetime satisfy

$$\left[\bar{X}^\mu, \bar{X}^\nu\right] = \mathrm{i}\bar{\theta}^{\mu\nu}, \tag{3.74}$$

where $\bar{\theta}^{\mu\nu}$ is *constant* and it is assumed to be non-degenerate. The fluctuations $\mathcal{A}^\mu$ can be described by covariant coordinates

$$X^\mu = \bar{X}^\mu + \mathcal{A}^\mu(\bar{X}), \tag{3.75}$$

where the matrices X^μ fulfill Eq. (3.74). We assume that the hermitian matrices $\mathcal{A}^\mu(\bar{X})$ can be interpreted at least locally as smooth functions on $\mathbb{R}^4_{\bar{\theta}}$, that is to say in the semi-classical limit we have

$$\mathcal{A}^\mu(\bar{X}) \sim \mathcal{A}^\mu(\bar{x}). \tag{3.76}$$

The matrices $\bar{X}^\mu$ are to be considered as the quantization of the coordinates $\bar{x}^\mu$ of Moyal-Weyl space. Notice that the commutation relations Eq. (3.74) satisfy the equation of motion

$$\left[\bar{X}^\mu, \left[\bar{X}^\nu, \bar{X}^{\mu'}\right]\right]\eta_{\mu\mu'} = 0. \tag{3.77}$$

Using the following change of variables

$$\mathcal{A}^\mu(\bar{x}) = -\bar{\theta}^{\mu\nu}A_\nu(\bar{x}) \tag{3.78}$$

we can write

$$\begin{aligned}
[X^\mu, f(X)] &= \left[\bar{X}^\mu + \mathcal{A}^\mu(\bar{X}), f(X)\right] \\
&= \mathrm{i}\bar{\theta}^{\mu\nu}\left(\partial_\nu + \mathrm{i}\left[A_\nu(\bar{x}), \right]\right) f(\bar{x}) \\
&\equiv \mathrm{i}\bar{\theta}^{\mu\nu}D_\nu f(\bar{x}).
\end{aligned} \tag{3.79}$$

This formula is exact up to boundary terms because

$$\left[\bar{X}^\mu, f(\bar{X})\right] = \mathrm{i}\,\bar{\theta}\partial_\mu f(\bar{X}) \tag{3.80}$$

is exact.

Scalars. The scalar action can now be written as

$$\begin{aligned} S[\Phi] &= -(2\pi)^2 \operatorname{Tr} [X^\mu, \Phi] [X^\nu, \Phi] g_{\mu\nu} \\ &= (2\pi)^2 \operatorname{Tr} \bar\theta^{\mu\alpha} \bar\theta^{\nu\beta} (D_\mu \Phi)(D_\nu \Phi) g_{\alpha\beta} \\ &= \int \mathrm{d}^4 x\, \bar\rho\, \bar g^{\mu\nu} (D_\mu \Phi)(D_\nu \Phi), \end{aligned} \tag{3.81}$$

where the effective geometry Eq. (3.34) for the Moyal-Weyl plane is indeed flat:

$$\bar g^{\mu\nu} = \bar\theta^{\mu\alpha} \bar\theta^{\nu\beta} g_{\alpha\beta}. \tag{3.82}$$

The symplectic volume factor is given by

$$\bar\rho = (\det \bar\theta^{\mu\nu})^{-1/2} = |\det \bar g_{\mu\nu}|^{1/4} \equiv \Lambda_{\mathrm{NC}}^4. \tag{3.83}$$

By defining the unimodular metric

$$\widetilde g^{\mu\nu} = \bar\rho\, \bar g^{\mu\nu}, \qquad |\det \widetilde g| = 1 \tag{3.84}$$

the action Eq. (3.81) becomes covariant

$$S[\Phi] = \int \mathrm{d}^4 \bar x\, \widetilde g^{\mu\nu} (D_\mu \Phi)(D_\nu \Phi). \tag{3.85}$$

In the geometrical interpretation of Eq. (3.81) - which took the form

$$S[\Phi] = \int \mathrm{d}^4 x \sqrt{|\widetilde G_{\mu\nu}|}\, \widetilde G^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) \tag{3.86}$$

- the $U(1)$ gauge field $A_\mu(x)$ is completely absorbed in the metric $\widetilde G^{\mu\nu}(x)$. Hence the $U(1)$ "photon" is actually a (gauge-fixed) graviton.

Gauge fields. Using covariant coordinates we can write the commutator $[X^\mu, X^\nu]$ as

$$\begin{aligned} [X^\mu, X^\nu] &= \left[\bar X^\mu + \mathcal{A}^\mu, \bar X^\nu + \mathcal{A}^\nu\right] \\ &= \left[\bar X^\mu, \bar X^\nu\right] + \left[\bar X^\mu, \mathcal{A}^\nu\right] - \left[\bar X^\nu, \mathcal{A}^\mu\right] + [\mathcal{A}^\mu, \mathcal{A}^\nu] \\ &= \bar\theta^{\mu\nu} + \mathcal{F}^{\mu\nu}, \end{aligned} \tag{3.87}$$

where

$$\begin{aligned} \mathcal{F}^{\mu\nu} &= \left[\bar X^\mu, \mathcal{A}^\nu\right] - \left[\bar X^\nu, \mathcal{A}^\mu\right] + [\mathcal{A}^\mu, \mathcal{A}^\nu] \\ &= -\mathrm{i}\, \bar\theta^{\mu\nu} \bar\theta^{\rho\sigma} \left(\partial_\nu A_\sigma - \partial_\sigma A_\nu - \mathrm{i}\, [A_\nu, A_\sigma]\right) \\ &= -\mathrm{i}\, \bar\theta^{\mu\nu} \bar\theta^{\rho\sigma} F_{\nu\sigma}. \end{aligned} \tag{3.88}$$

The corresponding noncommutative Yang-Mills action is then

$$\begin{aligned} S_{\mathrm{YM}} &= (2\pi)^2 \, \mathrm{Tr}\, [X^\mu, X^\nu]\, [X^\rho, X^\sigma]\, g_{\mu\rho}\, g_{\nu\sigma} \\ &= \int \mathrm{d}^4\bar{x}\, \bar{\rho}\, \big(\bar{\theta}^{\mu\nu} - \mathrm{i}\bar{\theta}^{\mu\alpha}\bar{\theta}^{\nu\beta}F_{\alpha\beta}\big)\big(\bar{\theta}^{\rho\sigma} - \mathrm{i}\bar{\theta}^{\rho\alpha'}\bar{\theta}^{\sigma\beta'}F_{\alpha'\beta'}\big) g_{\mu\rho}\, g_{\nu\sigma} \\ &= \int \mathrm{d}^4\bar{x}\, \bar{\rho}\, \Big(-\bar{g}^{\alpha\alpha'}\bar{g}^{\beta\beta'}F_{\alpha\beta}F_{\alpha'\beta'} + \bar{g}^{\alpha\beta}g_{\alpha\beta}\Big)\,. \end{aligned} \tag{3.89}$$

Thus we have shown that the matrix model action Eq. (3.1) and the scalar matrix model Eq. (3.11) can indeed be interpreted as a noncommutative gauge theory. In Chapter 5 we will show that this is also true for fermions. This result is no surprise since all three types of fields couple to the effective metric $\widetilde{G}^{\mu\nu}$.

3.2.2 Non-Abelian gauge fields

Non-Abelian gauge fields were studied first in [16] for the four-dimensional case and in [51] for the case of extra dimensions. Here we will only review the latter. Let us consider once more the matrix model

$$S_{\mathrm{YM}} = -\mathrm{Tr}\, [Y^a, Y^b][Y^{a'}, Y^{b'}]\eta_{aa'}\eta_{bb'}\,. \tag{3.90}$$

We want to understand general non-Abelian fluctuations around a Moyal-Weyl background. We have emphasized this by changing notation. As shown in [51] these fluctuations can be parameterized as follows

$$\begin{aligned} \begin{pmatrix} Y^\mu \\ Y^i \end{pmatrix} &= \begin{pmatrix} X^\mu \otimes \mathbb{1}_n + \mathcal{A}^\mu \\ \phi^i \otimes \mathbb{1}_n + \Phi^i + \mathcal{A}^\rho\partial_\rho(\phi^i \otimes \mathbb{1}_n + \Phi^i) \end{pmatrix} \\ &\sim (1 + \mathcal{A}^\rho\partial_\rho) \begin{pmatrix} X^\mu \otimes \mathbb{1}_n \\ \phi^i \otimes \mathbb{1}_n + \Phi^i \end{pmatrix}, \end{aligned} \tag{3.91}$$

where

$$\mathcal{A}^\mu = \mathcal{A}^\mu_\alpha \otimes \lambda^\alpha = -\theta^{\mu\nu} A_{\nu,\alpha} \otimes \lambda^\alpha \tag{3.92}$$

denote the $SU(N)$ gauge fields,

$$\Phi^i = \Phi^i_\alpha \otimes \lambda^\alpha \tag{3.93}$$

denotes $SU(N)$-valued scalar fields, and where λ^α are the representations of the $SU(N)$ gauge group. The covariant coordinates

$$X^\mu = \bar{X}^\mu + \mathcal{A}^{\mu,0} \tag{3.94}$$

denote the trace-$U(1)$ sector which was shown to describe gravity. Eq. (3.91) corresponds to the leading term in a Seiberg-Witten map [59] which relates commutative and noncommutative gauge theories with the appropriate gauge transformation. The effective action for $SU(N)$ gauge fields A_μ on $\mathcal{M}_\theta^{2n} \subset \mathbb{R}^D$ with $n = 2$ was derived in [51] to be

$$\begin{aligned} S_{\mathrm{YM}} \sim & \int \mathrm{d}^4x \, |\widetilde{G}_{\mu\nu}|^{1/2} \left(4\widetilde{\eta}(x) + e^\sigma \widetilde{G}^{\mu\mu'} \widetilde{G}^{\nu\nu'} \operatorname{tr}(F_{\mu\nu} F_{\mu'\nu'}) \right) \\ & + 2 \int \mathrm{d}^4x \, \eta(x) \operatorname{tr} F \wedge F, \end{aligned} \tag{3.95}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \mathrm{i}[A_\mu, A_\nu] \tag{3.96}$$

is the non-Abelian field strength tensor, and

$$F \wedge F = \frac{2}{4!} \left(F_{\mu\nu} F_{\rho\sigma} - F_{\mu\sigma} F_{\rho\nu} - F_{\sigma\nu} F_{\rho\mu} \right) \epsilon^{\mu\nu\rho\sigma}. \tag{3.97}$$

The action Eq. (3.95) coming from a $U(N)$ matrix model describes an action for a $SU(N)$ gauge field coupled to a dynamical metric $\widetilde{G}^{\mu\nu}(x)$ and the background metric $g_{\mu\nu}(x)$. This gives a new interpretation of the well-known fact that in noncommutative gauge theory the $U(1)$ and the $SU(N)$ groups cannot be disentangled. Here it has a natural explanation: Gravity couples to all fields. Especially, there exists a coupling of the effective metric $\widetilde{G}^{\mu\nu}$ - encoding the $U(1)$ gauge fields - to the field strength tensor of the $SU(N)$ fields.

In [51] it was shown that also the equations of motion for the non-Abelian gauge fields follow from the matrix Noether theorem

$$[Y^a, T^{bc}]\eta_{ab} = 0, \tag{3.98}$$

where T^{ab} is now given in terms of Y-matrices

$$T^{ab} = [Y^a, Y^c][Y^b, Y^d]\eta_{cd} + [Y^b, Y^c][Y^a, Y^d]\eta_{cd} - \frac{1}{2}\eta^{ab}[Y^c, Y^c][Y^{c'}, Y^{d'}]\eta_{cc'}\eta_{dd'}. \tag{3.99}$$

3.3 UV/IR mixing in NC emergent gravity

In this section we reveal how the Einstein-Hilbert action arises in noncommutative emergent gravity. As the name suggests, the Einstein-Hilbert action is not present at tree-level, rather it emerges at one-loop level of perturbation theory. This will shed new light on the severe problem of UV/IR mixing.

3.3.1 Emergent gravity

We claim that noncommutative $U(N)$ theory is actually an $SU(N)$ gauge theory coupled to gravity. So far we have shown how scalar fields and $SU(N)$ gauge fields couple in covariant manner to the effective metric $\widetilde{G}^{\mu\nu}$. It remains to be explained how the Einstein-Hilbert action

$$S_{\rm EH} = \int \mathrm{d}^4x \sqrt{|\widetilde{G}|}\, R[\widetilde{G}] \tag{3.100}$$

or some suitable alternative appears in this framework. It does not seem feasible to add an appropriate term in the matrix model action where the only objects existing are the matrices X^a. We argue that this is not necessary anyways because the Einstein-Hilbert action will arise automatically *upon quantization*. This is the idea of *emergent* or *induced gravity* which was put forward first by Sakharov [60]. A more recent discussion is given in [61].

Let us study the quantization of the matrix scalar model for simplicity. In principle, the quantization is defined in terms of a "path integral" over all matrices X^μ and Φ. In four dimensions, we can only perform perturbative computations for the "gauge sector" encoded by the matrices X^μ, while the scalars can be integrated out formally via a functional determinant

$$e^{-\Gamma_\Phi} = \int \mathrm{d}\Phi\, e^{-S[\Phi]}. \tag{3.101}$$

For non-interacting scalar fields this is given by

$$\Gamma_\Phi = \frac{1}{2} \mathrm{Tr} \log \frac{1}{2} \Delta_{\widetilde{G}}, \tag{3.102}$$

where $\Delta_{\widetilde{G}}$ is the Laplace-Beltrami operator of a scalar field on a classical Riemannian manifold $(\mathcal{M}, \widetilde{G}^{\mu\nu}(x))$ with the action Eq. (3.19). We express the functional determinant via an integral representation

$$\begin{aligned} \mathrm{Tr}\left(\log \frac{1}{2}\Delta_{\widetilde{G}}\right) &= -\mathrm{Tr} \int_0^\infty \frac{\mathrm{d}\alpha}{\alpha} \left(e^{-\frac{\alpha}{2}\Delta_{\widetilde{G}}}\right) \\ &\equiv -\mathrm{Tr} \int_0^\infty \frac{\mathrm{d}\alpha}{\alpha} \left(e^{-\frac{\alpha}{2}\Delta_{\widetilde{G}}}\right) e^{-\frac{1}{\alpha\Lambda^2}}. \end{aligned} \tag{3.103}$$

We have also introduced a UV cutoff Λ which regularizes the divergence for small α. Now we can apply the well-known heat kernel expansion [62, 63]

$$\mathrm{Tr} e^{-\frac{\alpha}{2}\Delta_{\widetilde{G}}} \sim \sum_{n \geq 0} \left(\frac{\alpha}{2}\right)^{\frac{n-4}{2}} \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{|\widetilde{G}_{\mu\nu}|}\, a_n(x, \Delta_{\widetilde{G}}). \tag{3.104}$$

The objects $a_n(x, \Delta_{\widetilde{G}})$ are known as Seeley-de Witt coefficients, which for the scalar action Eq. (3.19) are given by [62, 63]

$$\begin{aligned} a_0(x) &= \frac{1}{16\pi^2}, \\ a_2(x) &= \frac{1}{16\pi^2}\left(\frac{1}{6}\, R[\widetilde{G}]\right), \\ a_3(x) &= \frac{1}{16\pi^2}\frac{1}{360}\left(12 R_{;\mu}{}^{\mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \end{aligned} \tag{3.105}$$

The one-loop effective action is thus obtained as

$$\Gamma_\Phi = \frac{1}{16\pi^2} \int \mathrm{d}^4x \sqrt{|\widetilde{G}_{\mu\nu}|} \left(-2\Lambda^4 - \frac{1}{6} R[\widetilde{G}]\Lambda^2 + O(\log \Lambda) \right). \tag{3.106}$$

$R[\widetilde{G}]$ is the Ricci scalar with respect to the effective metric $\widetilde{G}_{\mu\nu}$. The Einstein-Hilbert action is indeed induced upon quantization.

A few remarks are in order. First, notice that the term $\int \mathrm{d}^4x \sqrt{|\widetilde{G}_{\mu\nu}|}\, \Lambda^4$ corresponds to a cosmological constant term in general relativity. Since the cutoff Λ is large, this term is usually in dramatic conflict with observation. We will show that this problem might be resolved in our framework, see Sect. 3.5. Secondly, Eq. (3.106) relates the cutoff to Newton's constant

$$\Lambda^2 \sim \frac{1}{G}. \tag{3.107}$$

In emergent gravity, gravitational interaction is weak if the cutoff is large. This should be the case here since Λ should be related to the scale of $\mathcal{N} = 4$ supersymmetry breaking as will be argued in Sect. 3.4.

In general relativity, the Einstein equation which encodes the dynamic of spacetime is the equation of motion obtained from varying the Einstin-Hilbert action with respect to the metric. Here the circumstances are somewhat different since the dynamics of $\theta^{\mu\nu}$ and ϕ^i are already determined at tree level by the corresponding e.o.m. (3.45) and (3.52). Moreover, at one-loop level the variation of the Einstein-Hilbert action is *not* with respect to the metric $\widetilde{G}_{\mu\nu}$ since this is not a fundamental object of the theory, rather the variation is with respect to $\theta^{\mu\nu}$ and ϕ^i. However, the tree level and the one-loop level conditions should be consistent. This is true at least as long as matter is not coupled to the model due to the fact that the equations of motion follow from a Noether theorem, see Sect. 3.1.4. How this changes once matter is coupled has not been worked out so far.

3.3.2 A new interpretation of UV/IR mixing

In this section we come back to the dual interpretation of our model as either a theory of gravity or a gauge theory. We begin with the "Moyal-Weyl point of view", that is we study a noncommuative gauge theory on Moyal-Weyl space with the action

$$\begin{aligned} S[\Phi] &= \frac{1}{2} \int \mathrm{d}^4\bar{x}\, \bar{\rho}\, \left(\bar{g}^{\mu\nu}(D_\mu\Phi)(D_\nu\Phi)\right) \\ &= \frac{1}{2} \int \mathrm{d}^4\bar{x}\, \widetilde{g}^{\mu\nu}(D_\mu\Phi)(D_\nu\Phi) \\ &= \frac{1}{2} \int \mathrm{d}^4\bar{x}\, \left(\Phi\Delta_A\Phi\right), \end{aligned} \tag{3.108}$$

Figure 3.1: Contributing Feynman diagrams to the one-loop effective action for a scalar field coupled to a $U(1)$ gauge field on four-dimensional Moyal-Weyl space.

where

$$\bar{\rho} = (\det \bar{\theta}^{\mu\nu})^{-1/2}, \tag{3.109}$$

as well as

$$\bar{g}^{\mu\nu} = \bar{\theta}^{\mu\alpha}\bar{\theta}^{\nu\beta}\eta_{\alpha\beta}, \quad \widetilde{g}^{\mu\nu} = \bar{\rho}\,\bar{g}^{\mu\nu}, \quad \text{and} \quad \Delta_A = -\widetilde{g}^{\mu\nu} D_\mu D_\nu. \tag{3.110}$$

The metric $\widetilde{g}^{\mu\nu}$ defined in Eq. (3.84) is unimodular. D_μ denotes the covariant derivative with a $U(1)$ gauge field

$$D_\mu = \partial_\mu + \mathrm{i}\, g\, [A_\mu, .]. \tag{3.111}$$

g is the coupling constant. In [48] the one-loop effective action of this action was computed by integrating out the scalar field Φ,

$$e^{-\Gamma_\Phi} = \left\langle \exp\left(- \int \mathrm{d}^4\bar{x} \left(\mathrm{i}g\partial_\mu\Phi[A_\nu, \Phi]\widetilde{g}^{\mu\nu} - \frac{g^2}{2}[A_\mu, \Phi][A_\nu, \Phi]\widetilde{g}^{\mu\nu} \right)\right)\right\rangle. \tag{3.112}$$

The contributing diagrams are depicted in Fig. 3.1. It was shown that this effective action is given by

$$\begin{aligned}\Gamma_\Phi = -\frac{g^2}{2}\frac{1}{16\pi^2} \int \mathrm{d}^4\bar{x} \Bigg(&\frac{\Lambda^4}{4}(\bar{\theta}_{\mu\nu}F^{\mu\nu})^2 + \bar{\rho}\frac{\Lambda^2}{24}\big(F_{\mu\nu}\partial^\lambda\partial_\lambda F_{\rho\sigma}\bar{g}^{\rho\mu}\bar{g}^{\sigma\nu} \\ &+ (\bar{\theta}_{\mu\nu}F^{\mu\nu})\partial^\rho\partial_\rho(\bar{\theta}_{\alpha\beta}F^{\alpha\beta})^2\big) + O(\log(\Lambda)) + O(A^3)\Bigg).\end{aligned} \tag{3.113}$$

This can now be compared to the one-loop effective action coming from the geometrical picture

$$\Gamma_\Phi = \frac{1}{16\pi^2} \int \mathrm{d}^4x \left(-2\Lambda^4 - \frac{1}{6}R[\widetilde{G}]\Lambda^2 + O(\log(\Lambda)) \right). \tag{3.114}$$

The Ricci scalar can be written in terms of the effective metric $\widetilde{G}_{\mu\nu}$ which itself is composed by the Poisson structure $\theta^{\mu\nu}$. We can express the Poisson tensor in terms of the $U(1)$ field strength by

$$[X^\mu, X^\nu] = \mathrm{i}\theta^{\mu\nu}(X) = \mathrm{i}\bar{\theta}^{\mu\nu} - \mathrm{i}\bar{\theta}^{\mu\alpha}\bar{\theta}^{\nu\beta}F_{\alpha\beta}. \tag{3.115}$$

Thus the effective metric is given by

$$\widetilde{G}^{\mu\nu}(x) = (\bar{\theta}^{\mu\alpha} - \bar{\theta}^{\mu\rho}\bar{\theta}^{\alpha\sigma}F_{\rho\sigma})(\bar{\theta}^{\nu\beta} - \bar{\theta}^{\nu\kappa}\bar{\theta}^{\beta\lambda}F_{\kappa\lambda})\eta_{\alpha\beta}. \tag{3.116}$$

One can rewrite the Ricci scalar now in terms of the field strength tensor. The result can be shown [48] to match precisely Eq. (3.113) which, however, suffers from UV/IR mixing[6]. This gives the mixing problem a new interpretation: It is an effect of gravity. The UV divergences of the commutative theory are turned into infrared gravitational effects by the noncommutativity of the theory. The geometrical picture turns out to be valid for momenta $p\,\Lambda < \Lambda_{\mathrm{NC}}^4$. This new interpretation does not by itself render the theory renormalizable which means that there really should be a cutoff $\Lambda \leq \Lambda_{\mathrm{NC}}$. It does, however, give an explanation why the UV/IR mixing is restricted to the $U(1)$ sector of the $U(N)$ gauge theory: Only $U(1)$ degrees of freedom contribute to the metric.

3.4 A model for quantum gravity? - Relations to string theory

In this section we want to discuss how the matrix model Eq. (3.1) is understood from a string theory point of view. We will also discuss why the IKKT model is singled out as prime candidate for a theory of quantum gravity in the context of noncommutative emergent gravity.

The relations between noncommutative geometry and string theory are deep and they have been studied intensively at the end of the 1990ies, see e.g. [55, 42, 59, 64]. Matrix models are of crucial relevance because they describe on the one hand - as we have seen in Sect. 2.4 - noncommutative Yang-Mills theory and on the other hand they are conjectured to define non-perturbative Type IIB string dynamics. Central in that context is the IKKT model which is stated in ten spacetime dimensions and whose action is given by

$$S_{\mathrm{IKKT}} = -\mathrm{Tr}\left(\frac{1}{4}\left[Y^a, Y^b\right]\left[Y^{a'}, Y^{b'}\right]\eta_{aa'}\eta_{bb'} + \frac{1}{2}\bar{\Psi}\gamma^a\left[Y_a, \Psi\right]\right) \qquad a, b = 0, \ldots 10. \tag{3.117}$$

Here Ψ is a ten-dimensional Majorana-Weyl fermion, and Y^a and Ψ are $N \times N$ hermitian matrices. It can be shown [55] that the Green-Schwarz action of type IIB superstring theory on flat ten-dimensional space $\mathbb{R}^{10}$ - which in Schild gauge writes as

$$S_{\mathrm{Schild}} = \int \mathrm{d}^2\sigma\sqrt{|g|}\left(\frac{\alpha}{4}\left\{X^a, X^b\right\}^2 - \frac{\mathrm{i}}{2}\bar{\Psi}\gamma^a\left\{X^a, \Psi\right\} + \beta\right) \tag{3.118}$$

- can be regarded as a classical limit of the above model. The IKKT model was proposed to be a non-perturbative formulation of Type IIB superstring theory [42].

[6]For the sake of simplicity we have suppressed some subtleties here. These will be discussed in detail in Chapter 5.

This matrix model comes with a $\mathcal{N}=2$ supersymmetry [55]

$$\begin{aligned} \delta^{(1)}\Psi &= \frac{\mathrm{i}}{2}[Y^a, Y^b]\gamma_{ab}\,\epsilon, & \delta^{(1)}Y^a &= \mathrm{i}\,\bar{\epsilon}\,\gamma^a\,\Psi, \\ \delta^{(2)}\Psi &= \xi, & \delta^{(2)}Y^a &= 0, \end{aligned} \tag{3.119}$$

where ϵ and ξ are Grassmann-valued spinors and $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$. It is worthwhile mentioning that the IKKT model itself is a large N reduced model of ten-dimensional $\mathcal{N}=2$ super Yang-Mills theory. The ten-dimensional $U(N)$ super Yang-Mills action is given by

$$S_{\mathrm{SYM}} = \frac{1}{4}\int \mathrm{d}^{10}x\,\mathrm{Tr}\left(F_{ab}F^{ab} + 2(\bar{\Psi}\gamma^a\,D_a\Psi)\right), \tag{3.120}$$

where the Dirac operator is given by

$$D_a\Psi = \partial_a\Psi - \mathrm{i}[A_a, \Psi]. \tag{3.121}$$

When reduced to a point p the integral is redundant and the remaining field strength tensor is

$$\begin{aligned} F_{ab} &= \partial_a A_b(x)|_p - \partial_b A_a(x)|_p + \mathrm{i}[A_a, A_b]|_p \\ &= \mathrm{i}[A_a(p), A_b(p)]. \end{aligned} \tag{3.122}$$

Performing the large N-limit and identifying the gauge field $A_a(p)$ of the $\mathcal{N}=2$ super Yang-Mills theory with the matrices Y^a of Eq. (3.117) shows that the action Eq. (3.120) gives indeed the IKKT model Eq. (3.117).

What makes the IKKT model so exciting is the fact that it is also related to $\mathcal{N}=4$ super Yang-Mills theory in *four* spacetime dimensions. The excitement is due to the conjecture that $\mathcal{N}=4$ super Yang-Mills theory is *finite*. Although a rigorous proof is still missing and it also does not seem feasible in the near future due to enormous technical difficulties there exist indications that this theory might be finite. Firstly, the beta-function in conventional, i.e. commutative, $\mathcal{N}=4$ super Yang-Mills theory is vanishing [65, 66]. Secondly the corresponding noncommutative model was shown [67, 68] not to be plagued by UV divergences which could lead to UV/IR mixing. It is thus plausible that UV/IR mixing is not present at all and the theory might be finite even in the noncommutative case. The IKKT model is hence a good candidate for a theory of quantum gravity. From the point of view of noncommutative emergent gravity, it is a prime candidate.

The finiteness of noncommutative $\mathcal{N}=4$ super Yang-Mills theory is of fundamental importance for noncommutative emergent gravity. It is a "working hypothesis" so to speak. As we have seen, noncommutative emergent gravity gives a beautiful new interpretation to UV/IR mixing. However, it does not make the theory renormalizable by that. The cutoff we have introduced in Sect. 3.3.2 really should be regarded as a physical cutoff and it should be given by the scale of $\mathcal{N}=4$ supersymmetry breaking.

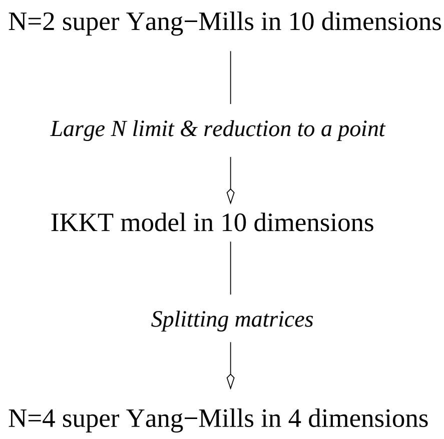

Figure 3.2: The relations between $\mathcal{N}=2$ and $\mathcal{N}=4$ super Yang-Mills theories in ten and four dimensions and the IKKT model.

Above this scale this theory is consequently assumed to be finite according to our working hypothesis.

$\mathcal{N}=4$ super Yang-Mills theory is recovered from the IKKT model by splitting the matrices according to Sect. 3.1.3:

$$Y^a = \begin{pmatrix} Y^\mu \\ \Phi^i(Y^\mu) \end{pmatrix}, \tag{3.123}$$

where $\Phi^i(Y^\mu)$ correspond to the non-Abelian scalar fields of Eq. (3.93). This gives us finally our full motivation to consider the matrix model in ten dimensions. Our model Eq. (3.1) corresponds to the IKKT model if it contains $n_S = 6$ scalar fields (which it does) and $n_\Psi = 2$ Dirac fermions. Because $\mathcal{N}=4$ super Yang-Mills theory is expected not to know anything of UV/IR mixing [67] and because the induced action is the same whether it is obtained from the "NC gauge theory point of view"as in Sect. 3.3.1 or from the "geometrical point of view" as in Sect. 3.3.2 one finds

$$\Gamma_A = -2\Gamma_\Psi - 6\Gamma_\Phi, \tag{3.124}$$

where Γ_A is given by

$$e^{-\Gamma_A} = \int_{\text{one-loop}} \mathrm{d}A\, e^{-S}. \tag{3.125}$$

3.5 Physical solutions

In the previous sections we have given an introduction to noncommutative emergent gravity. We come now to explicit solutions of this framework. In a first part we ask how noncommutative emergent gravity differs from general relativity. Then we investigate solutions of noncommutative emergent gravity which are of Friedmann-Robertson-Walker type. Thereafter we briefly discuss solutions which describe spherical matter distributions within our universe. We follow references [53, 54].

3.5.1 A comparison of NC emergent gravity and general relativity

So far we have revealed that the matrix model Eq. (3.1) contains gravity. It remains to be discussed what kind of gravity theory this is. That is to say we have to clarify what the differences with respect to general relativity are.

In general relativity the physically realized spacetime geometries have to fulfill the equations of motion which are derived by varying the Einstein-Hilbert action. The variation is with respect to the metric which is the fundamental object in Einstein gravity. The obtained equations of motion are of course the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R\, \widetilde{G}_{\mu\nu} + \Lambda\, \widetilde{G}_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{3.126}$$

In noncommutative emergent gravity physically realized geometries have to fulfill equations of motion, as well. However, these equations are derived from the matrix model and they are found to be

$$\widetilde{G}^{\rho\sigma} \widetilde{\nabla}_\rho (e^\sigma \theta^{-1}_{\sigma\nu}) = e^{-\sigma} \widetilde{G}_{\mu\nu} \theta^{\mu\alpha} \partial_\alpha \eta(x), \tag{3.127}$$

where $\eta(x)$ was given in the last line of Eq. (3.38). Since this equation determines spacetime in this model it is in some sense the analogue of the Einstein equation. The Einstein-Hilbert action on the other hand emerges at one-loop level. Variations thereof have to be studied with respect to the Poisson structure $\theta^{\mu\nu}$ and the background metric $g_{\mu\nu}$, respectively the scalar fields ϕ^i. Since the above covariant equation of motion is the consequence of a Noether theorem it is protected from quantum corrections. Hence the equations of motion derived from the Einstein-Hilbert action should be consistent with the tree level equations and corrections are not expected. However, this is probably only true for a universe without matter. How these relations change when matter is coupled to the model has not been worked out so far.

In principle, the above considerations mean that we have found new equations of motion. This could be considered as a rather radical approach given the success of general relativity. However, the basic question is whether these new equations allow for solutions which are consistent with observation. As we will shown in the upcoming sections, the first studies have been very promising, but clearly a lot of work is required to test the model in detail and before a final answer can be given.

3.5.2 Self-dual solutions

An important class of solutions of the equations of motion (3.46) is given by two-forms θ^{-1} satisfying

$$\widetilde{G}_{\mu\nu} = g_{\mu\nu}, \tag{3.128}$$

i.e. the effective metric and the background metric are identical. In this case the covariant e.o.m. reduce to

$$g^{\mu\nu}\nabla_\mu \theta^{-1}_{\nu\alpha} = 0, \tag{3.129}$$

where ∇_μ is the Levi-Civita connection with respect to $g_{\mu\nu}$.

These are formally the free Maxwell equations in the background geometry $g_{\mu\nu}$. In four dimensions the condition Eq. (3.128) is equivalent to self-dual two-forms $\theta^{-1}(x)$. To see this for Euclidean signature[7] use appropriate $SO(4)$ transformations such that at a point x the induced metric takes the form $g_{\mu\nu} = \mathrm{diag}(1, 1, 1, 1)$ and $\theta^{\mu\nu}$ becomes

$$\sqrt{\rho}\,\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1} & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}. \tag{3.130}$$

This is (anti-)self-dual $\star\theta^{-1} = \pm\theta^{-1}$ if and only if $\alpha^2 = 1$, where $\star$ denotes the Hodge star, and $\theta^{-1} = \theta^{-1}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. The effective metric at the point x is

$$G^{\mu\nu} = \rho\,\theta^{\mu\alpha}\theta^{\nu\beta}g_{\alpha\beta} = \mathrm{diag}\left(\alpha^2, \alpha^{-2}, \alpha^{-2}, \alpha^2\right). \tag{3.131}$$

If θ^{-1} is (anti-)self-dual then $\widetilde{G}^{\mu\nu} = \mathrm{diag}(1, 1, 1, 1) = g^{\mu\nu}$. On the other hand if $\widetilde{G}^{\mu\nu} = g^{\mu\nu}$, then

$$\eta = \frac{1}{4}e^\sigma \widetilde{G}^{\mu\nu}\eta_{\mu\nu} = \frac{e^\sigma}{2}\left(\alpha^2 + \alpha^{-2}\right) = e^\sigma \tag{3.132}$$

and so $\alpha^2 = 1$ implying θ^{-1} is self-dual.

3.5.3 Cosmological solutions

Self-dual solutions are of special interest because they are related to the cosmological constant problem which was discussed in Chapter 1. We recall that the matrix model action in the semi-classical limit was given by Eq. (3.39)

$$S_{\mathrm{YM}} = \int \mathrm{d}^4x\,|\widetilde{G}_{\mu\nu}|^{1/2}\widetilde{G}^{\mu\nu}g_{\mu\nu}. \tag{3.133}$$

[7] For Minkowski space, i.e. $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, the $\sqrt{\rho}\,\theta^{03}$ component gets a factor i and the component $\sqrt{\rho}\,\theta^{12}$ a factor (-1). Then this two-form is i-self-dual, i.e. $\star\theta^{-1} = \mathrm{i}\theta^{-1}$, or anti-i-self-dual, $\star\theta^{-1} = -\mathrm{i}\theta^{-1}$.

For self-dual solutions this becomes

$$S_{\mathrm{YM}} = 4 \int \mathrm{d}^4x \, |g_{\mu\nu}|^{1/2}, \tag{3.134}$$

which is precisely the form of the induced vacuum energy interpreted as cosmological constant in general relativity. The variation of this term is given by

$$\delta \int \mathrm{d}^4x \sqrt{|g|} \sim \int \mathrm{d}^4x \, \sqrt{|g|} \, g^{\mu\nu} \delta g_{\mu\nu} = \int \mathrm{d}^4x \, \sqrt{|g|} \, (\Delta_g \phi^i) \, \delta\phi^j \, \delta_{ij}. \tag{3.135}$$

However, this term vanishes for on-shell geometries since they fulfill $\Delta_g \phi^i = 0$, see Sect. 3.1.4. The coefficient of this term is hence irrelevant and harmonically embedded geometries are protected from the cosmological constant problem. The term $\int \mathrm{d}^4x \sqrt{|g|}$ should therefore *not* be interpreted as a cosmological constant, rather it describes a brane tension. Therefore noncommutative emergent gravity might resolve the cosmological constant problem. However, this needs further studies since we have not coupled to matter yet.

Our universe. Our universe is a homogeneous and isotropic place if scales larger than 300 million lightyears [69] are considered, see Fig. 3.3. Moreover, Edwin Hubble [70] found in 1929 that the universe is expanding. The rate of this expansion is given by the "Hubble constant"[8]

$$H(t) = \frac{\dot{a}}{a}, \tag{3.136}$$

where $a(t)$ is the scaling factor of the universe. The present value of the Hubble constant is given by [71]

$$H_0 = 72 \pm 8 \; \frac{\mathrm{km}}{\mathrm{s \; Mpc}}. \tag{3.137}$$

Such a homogeneous and isotropic universe is described by the Friedmann-Robertson-Walker (FRW) metric

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{FRW}} &= -\mathrm{d}t^2 + a(t)^2 \left(\frac{1}{1-kr^2} \mathrm{d}r^2 + r^2 (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \right) \\ &= -\mathrm{d}t^2 + a(t)^2 \left(\frac{1}{1-kr^2} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \right). \end{aligned} \tag{3.138}$$

$k \in \{-1, 0, 1\}$ determines the curvature and hence the fate of the universe: $k = 0$ describes a flat universe that will expand forever, $k = 1$ corresponds to a closed universe which contains enough matter such that the universe will ultimately collapse in a "big

[8]Note well that the Hubble constant is not a constant.

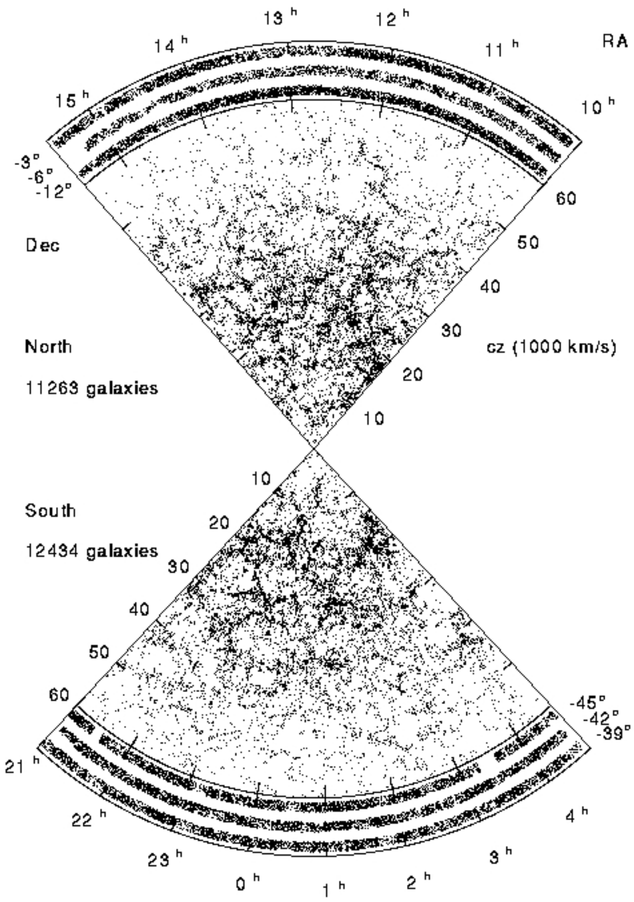

Figure 3.3: The Las Campanas Redshift Survey [69]. The redshifts of 23,697 galaxies were measured. The galaxies are clumped, with great voids, and great clusters, all on scales less than about 300 million lightyears. On larger scales, the structures even out - the universe becomes homogeneous and isotropic.

crunch", and $k = -1$ describes a hyperbolically curved, open universe that will expand forever, as well. For an introduction to cosmology see, for example, [72, 14].

The currently accepted model for cosmology is the so-called ΛCDM model. Λ stands for the cosmological constant or better "dark energy" which is introduced to explain the accelerating expansion of the universe [12, 13]. It should account for $\sim 74\%$ of the energy density in the universe. CDM is an abbreviation for Cold Dark Matter which is assumed to be a non-relativistic (i.e. cold), and non-baryonic form of matter that interacts almost only through gravity. Hence it cannot be seen. It is clearly not made of any particle contained in the Standard Model. 22% of the energy density of the universe are attributed to dark matter. The remaining 4% is visible matter, i.e. galaxies, stars, planets, and gas clouds. Neither the origin of dark energy nor of dark matter is explained in this model. The ΛCDM model assumes inflation [73] which is an exponential expansion phase at the beginning of the universe leading to a spatially flat universe, i.e. $k = 0$. After the end of inflation the dynamcis of the universe is governed by the Friedmann equations which follow from the Einstein equations for a FRW background,

$$\begin{aligned} H^2 = \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + p) + \frac{\Lambda}{3}, \end{aligned} \tag{3.139}$$

and the equation of state

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p), \tag{3.140}$$

where ρ is the energy density and p is the pressure of the matter content of the universe, and G denotes Newton's constant.

A possible alternative to the ΛCDM model. Let us now study cosmological solutions within the framework of noncommutative emergent gravity. If we want to find a (near-) realistic description of the universe which is not plagued by the cosmological constant problem we need to find an harmonic embedding of the Friedmann-Robertson-Walker metric. The following appropriate embedding for $k = \pm 1$ was found in [53]

$$\vec{x}(t, \chi, \theta, \varphi) = \begin{pmatrix} \mathcal{R}(t) \begin{pmatrix} S(\chi)\sin\theta\cos\varphi \\ S(\chi)\sin\theta\sin\varphi \\ S(\chi)\cos\theta \\ C(\chi) \end{pmatrix} \\ 0 \\ x_c(t) \end{pmatrix} \in \mathbb{R}^{10}, \tag{3.141}$$

where

$$\mathcal{R}(t) = a(t)\begin{pmatrix} \cos\psi(t) \\ \sin\psi(t) \end{pmatrix}, \tag{3.142}$$

$S(\chi) = (\sin\chi, \sinh\chi) = r$, and $C(\chi) = (\cos\chi, \cosh\chi)$ for $k = (1, -1)$, respectively. Moreover $\eta_{ab} = \mathrm{diag}(+, \ldots, \pm, -)$ for $k = 1$, or $\eta_{ab} = \mathrm{diag}(+ \ldots, +, -, -, +, +)$ for $k = -1$. For the FRW line element one finds

$$\mathrm{d}s^2_{\mathrm{FRW}} = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -c(t)\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}\chi^2 + S(\chi)^2\mathrm{d}\Omega^2\right) \tag{3.143}$$

with

$$c(t) = k(\dot{x}_c^2 - a^2\dot{\psi}^2 - \dot{a}^2). \tag{3.144}$$

The condition for FRW is then

$$\dot{x}_c^2 - a^2\dot{\psi}^2 - \dot{a}^2 = k. \tag{3.145}$$

The above embedding should be harmonic, i.e. it should fulfill the e.o.m. (3.52) and (3.53):

$$\Delta_g x^a = 0, \tag{3.146}$$

where we demand $\widetilde{G}_{\mu\nu} = g_{\mu\nu}$. For reasons of symmetry it is enough to evaluate

$$\begin{aligned} \Delta_g(R(t)S(\chi)\cos\theta) &\stackrel{!}{=} 0, \\ \Delta_g x_c &\stackrel{!}{=} 0. \end{aligned} \tag{3.147}$$

These conditions lead to three differential equations for $\psi(t)$, $a(t)$, and $x_c(t)$,

$$\frac{3}{a}\left(\dot{a}^2 + k\right) + \ddot{a} - \dot{\psi}^2 a = 0, \tag{3.148}$$

$$5\dot{\psi}\dot{a} + \ddot{\psi}a = 0, \tag{3.149}$$

$$\frac{3}{a}\dot{a}\dot{x}_c + \ddot{x}_c = 0. \tag{3.150}$$

The first two equations can be integrated which gives

$$H^2 = -b^2a^{-10} + m^{-8} - \frac{k}{a^2}, \tag{3.151}$$

$$\frac{\ddot{a}}{a} = -2ma^{-8} + 4b^2a^{-10}, \tag{3.152}$$

where $m > 0$, and b are integration constants whose physical meaning is not entirely understood, see below. Eq. (3.150) integrated is in fact a consequence of Eq. (3.145). These equations look somewhat like the Friedmann equations Eq.(3.139) for an empty universe without cosmological constant, and with $b = m = 0$

$$\begin{aligned} \dot{a}^2 + k &= 0, \\ \ddot{a} &= 0. \end{aligned} \tag{3.153}$$

However, the origin of Eq. (3.151) and (3.152) is completely different. They are obtained from purely geometrical considerations, i.e. from the harmonic embedding condition. The Friedmann equations on the other hand are derived from the Einstein equation of general relativity for a Friedmann-Robertson-Walker background.

From Eq. (3.151) one can read off immediately that for late times, i.e. $a(t)$ sufficiently large so that we can neglect $-b^2a^{-10}+ma^{-8}$, Eq. (3.151) gives

$$\dot{a} \sim -k. \tag{3.154}$$

For the case $k=1$ one finds an unrealistic since too short age of the universe given the present Hubble parameter. Therefore we study the case $k=-1$. Then we obtain a simple relation for the scale parameter

$$a(t) \sim t, \tag{3.155}$$

which describes the *Milne* universe [74].

In oder to avoid the cosmological constant problem we not only need an harmonic embedding, we also need a Poisson structure such that $\widetilde{G}_{\mu\nu}=g_{\mu\nu}$. In order to achieve that we introduce a new coordinate $\tilde{t}(t)$ via

$$\frac{\mathrm{d}\tilde{t}}{\tilde{t}} = \frac{\mathrm{d}t}{a}, \tag{3.156}$$

so that the FRW line element becomes

$$\mathrm{d}s^2_{\mathrm{FRW}} = \frac{a^2}{\tilde{t}^2}(-\mathrm{d}\tilde{t}^2+\tilde{t}^2\mathrm{d}\chi^2+\tilde{t}^2\sinh^2\mathrm{d}\Omega^2). \tag{3.157}$$

Another change of variables

$$\tau = \tilde{t}\cosh\chi, \qquad r=\tilde{t}\sinh\chi \tag{3.158}$$

shows that the FRW line element is conformally flat

$$\mathrm{d}s^2_{\mathrm{FRW}} = \frac{a^2}{\tilde{t}^2}\left(-\mathrm{d}\tau^2+\mathrm{d}r^2+r\mathrm{d}\Omega^2\right). \tag{3.159}$$

For large times the scale parameter evolves linearly in t, i.e. $a(t)\sim t$, and so the line element approaches flat Minkowski spacetime. An appropriate two-form θ^{-1} is then given by

$$\theta^{-1} = \theta^{-1}_{\mu\nu}\,\mathrm{d}x^\mu\wedge x^\nu = \mathrm{i}\mathrm{d}\tau\wedge\mathrm{d}x^1+\mathrm{d}x^2\wedge x^3, \tag{3.160}$$

which is i-self-dual[9], i.e. $\star\theta^{-1}=\mathrm{i}\theta^{-1}$, and fulfills $\widetilde{G}_{\mu\nu}=g_{\mu\nu}$ as well as

$$|g_{\mu\nu}| = \frac{a^8}{\tilde{t}^8}, \quad |\theta^{-1}_{\mu\nu}|=1, \quad e^{-\sigma}=\frac{\tilde{t}}{a}. \tag{3.161}$$

[9]This is due to the Lorentzian signature of the Friedmann-Robertson-Walker metric.

A first confrontation with experiment. The Milne universe $a(t) = t$ is in remarkably good agreement with observations from type Ia supernovae[10] [75, 76] without the need to introduce any form of "dark energy". The age of the universe is predicted correctly and it is given by the inverse of the present Hubble constant

$$t_0 = \frac{1}{H_0} \approx 14 \times 10^9. \tag{3.162}$$

In the ΛCDM model one needs fine tuning such that the present value of the inverse Hubble constant gives approximately the correct age of the universe.

Early universe. So far we have only considered late times, now we want to consider the early universe as predicted by our model. The scaling parameter is determined by

$$\dot{a} = \sqrt{-b^2 a^{-10} + m a^{-6} + 1}. \tag{3.163}$$

For $b \neq 0$, we denote with a_0 the positive root of the argument, which corresponds to the "minimal size" of the universe. Thus in this model we do not find a "big bang", i.e. a singularity at the beginning of the universe, but rather a "big bounce". However, this conclusion could change once we couple matter to the model [77]. We fix the origin of time by $a(t = 0) = a_0$ and define t_1 by

$$\begin{aligned} \ddot{a}(t_1) &= 0, \\ a(t_1) &= \sqrt{\frac{4b^2}{3m}}, \end{aligned} \tag{3.164}$$

using Eq. (3.152). By expanding the expression $-b^2 a^{-8} + m a^{-6} + 1$ around a_0

$$\dot{a} \sim \sqrt{p(a - a_0)}, \qquad a(t) \sim \frac{p}{4} t^2 + a_0, \tag{3.165}$$

where

$$p = \frac{\mathrm{d}\left(-b^2 a^{-8} + m a^{-6} + 1\right)}{\mathrm{d}a}\bigg|_{a=a_0}, \tag{3.166}$$

one can observe that at the beginning of the universe there was a mild inflationary-like phase $\ddot{a} > 0$ which ended at

$$a(t_1) = \sqrt{\frac{4b^2}{3m}} \tag{3.167}$$

meaning that b and m are related to the end of this inflationary-like phase. A typical evolution of $a(t)$ and the Hubble parameter is shown in Fig. 3.4 for exemplary values

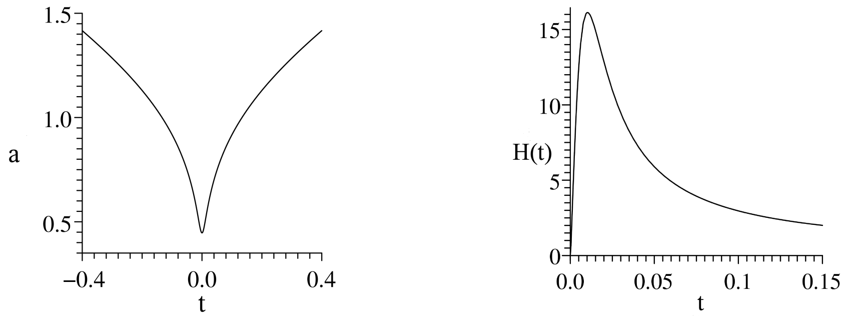

Figure 3.4: Evolution of the scale parameter $a(t)$ of the Friedmann-Robertson-Walker metric and of the Hubble constant $H(t)$ for $m = 5$, and $b = 1$. For large t the scale parameter evolves as the Milne universe [75, 76]. In the early universe we find a mild inflationary-like phase as well as a "minimal size" for the universe at $t = 0$ corresponding to a "big bounce" rather than a big bang.

$m = 5$ and $b = 1$. As it can also be seen in Fig. 3.4 for $b > 0$ the time evolution should in fact not start at $t = 0$, but be it can be completed symmetrically as

$$a(-t) = a(t), \qquad \psi(-t) = -\psi(t), \qquad x_c(-t) = -x_c(t). \tag{3.168}$$

In order to answer the question whether this model could be realized in nature, a more detailed analysis is required. In a first step matter should be coupled to the model. A crucial test would certainly be the computation of matter density fluctuations in the early universe that can be measured indirectly via temperature fluctuations in the cosmic microwave background [78, 14].

3.5.4 Solutions describing mass distributions

Clearly, it is of great importance to find physical solutions which describe spherical matter distributions[11] and which are thus an analog of the Schwarzschild solution of general relativity. If we want to study matter distributions within the universe we need to know the equations of motion for gravity coupled to matter. We assume once more

[10]A type Ia supernova is the explosion of an old star whose fusion process has come to an end. Such are star is called a "white dwarf" and its mass is limited to masses that are a little below the Chandrasekhar limit of about 1.38 solar masses. This category of supernovae produces consistent peak luminosity because of the uniform mass of white dwarfs around this limit. Due to the stability of the luminosity these explosions are used as standard candles in astronomy to measure their distances and redshifts.

[11]We follow [54] in this section.

$\widetilde{G}_{\mu\nu} = g_{\mu\nu}$. The total action in that case is

$$\begin{aligned} S &= S_{\text{vac}} + S_{\text{EH}} + S_{\text{matter}} \\ &= \int \mathrm{d}^4x \sqrt{|g|} \left(-2\Lambda_{\text{vac}}^4 + R[g]\Lambda_{\text{EH}}^2\right) + S_{\text{matter}}, \end{aligned} \tag{3.169}$$

where the vacuum term

$$S_{\text{vac}} = -2 \int \mathrm{d}^4x \sqrt{|g|} \Lambda_{\text{vac}}^4 \tag{3.170}$$

is the total contribution from bare terms coming from Yang-Mills theory and induced terms. This explains the signs in Eq. (3.169) and (3.170). The variation of Eq. (3.169) is given by

$$\delta S = \int \mathrm{d}^4x \sqrt{|g|}\, \mathcal{H}^{\mu\nu}\, \delta g_{\mu\nu}, \tag{3.171}$$

where

$$\mathcal{H}^{\mu\nu} = 8\pi T^{\mu\nu} - \Lambda_{\text{EH}}^2\, \mathcal{G}^{\mu\nu} - \Lambda_{\text{vac}}^4\, g^{\mu\nu} \quad \text{and} \quad \mathcal{G}^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R. \tag{3.172}$$

We have to vary the action Eq. (3.169) with respect to the embedding field ϕ^i since it is a fundamental object of the theory. This variation is up to boundary terms

$$\delta S = -2 \int \mathrm{d}^4x (\delta\phi^i) \partial_\mu (\sqrt{|g|} \mathcal{H}^{\mu\nu} \partial_\nu \phi^j) \delta_{ij}. \tag{3.173}$$

The equation of motion is thus

$$\partial_\mu (\sqrt{|g|} \mathcal{H}^{\mu\nu} \partial_\nu \phi^i) = 0, \tag{3.174}$$

which can be written as

$$\Lambda_{\text{vac}}^4\, \Delta_g \phi^i = (8\pi T^{\mu\nu} - \Lambda_{\text{EH}}^2 \mathcal{G}^{\mu\nu}) \nabla_\mu \partial_\nu \phi^i + 8\pi (\nabla_\mu T^{\mu\nu}) \partial_\nu \phi^i \tag{3.175}$$

using

$$\nabla_\mu V^\mu \equiv \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu), \quad \text{and} \quad \nabla_\mu \mathcal{G}^\mu = 0. \tag{3.176}$$

One of the central observations is the existence of two types or "branches" of solutions of this equation of motion: The "Einstein branch" and the "harmonic branch". Every solution of the Einstein equations is of course a solution of $\mathcal{H}^{\mu\nu} = 0$ and solves hence Eq. (3.174). This is the Einstein branch which is likely to be plagued by the same cosmological constant problem as general relativity. The harmonic branch gives solutions with

$$\partial_\mu (\sqrt{|g|} \mathcal{H}^{\mu\nu} \partial_\nu \phi^i) = 0, \qquad \mathcal{H}^{\mu\nu} \neq 0. \tag{3.177}$$

These are solutions where the matter density is negligible compared to the vacuum energy density and hence Eq. (3.175) reduces to $\Delta_g \phi^i = 0$. As discussed in the previous section, these solutions describe minimal surfaces that will be deformed locally in the presence of matter. This is a good class of solutions since it might avoid the cosmological constant problem.

In [54] static metrics $g_{\mu\nu}$ which are supposed to describe localized matter distributions and the Newtonian limit have been studied. The main result is that larger structures such as galaxies are embedded in so-called "gravity bags" which are deformations of the embedding $\mathcal{M}_\theta \in \mathbb{R}^{10}$ of the form

$$x^a = \begin{pmatrix} x^\mu \\ \phi^i \end{pmatrix} = \begin{pmatrix} t \\ x^i \\ g(x) \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} \end{pmatrix}, \tag{3.178}$$

where $g(x)$ is independent of t and $g(x) \to 0$ as $r \to \infty$. This is a rotating embedding with radial frequency ω. Due to its flat asymptotics, this type of embedding fits naturally to the cosmological embedding. Moreover, this ansatz leads to a static metric

$$\mathrm{d}s^2 = -(1 - \omega^2 g^2)\mathrm{d}t^2 + (\delta_{ij} + (\partial_i g)(\partial_j g))\mathrm{d}x^i \mathrm{d}x^j. \tag{3.179}$$

The e.o.m. for these gravity bags are for valid for empty space, hence $\rho = 0$, and for negligible curvature contributions. So they reduce to

$$\Delta_g \phi^i(x) = 0. \tag{3.180}$$

The unique spherically symmetry solution regular at the origin was found [54] to be

$$\begin{aligned} g_0(r) &= g_0 \frac{\sin(\omega r)}{\omega r} \\ \phi_0(x) &= g_0(r) e^{\mathrm{i}\omega t} = g_0(r) \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}. \end{aligned} \tag{3.181}$$

The effective gravitational potential of this gravity bag was determined as

$$U_0(r) \sim -\frac{\omega^2}{r^2}. \tag{3.182}$$

The gravitational force due to the potential $U_0(r)$ is rapidly decreasing with range $L_\omega = \frac{\pi}{\omega}$ which is in contradiction to Newton's law which prescribes a $1/r$ behaviour for the gravitational potential. However, the solution to this contradiction is given as follows.

The potential of these gravity bags given by Eq. (3.182) will lead to an attractive gravitational force so that matter will be accumulated within these bags with $\rho \neq 0$. It can be shown [54] that the matter within this region will deform the embedding such that this matter will experience Newtonian gravity. The physical situation one should think of in this context is a galaxy which will provide the gravity bag. The galaxy is an accumulation of stars which will then experience Newtonian gravity.

The embedding responsible for Newtonian gravity is given by

$$\begin{aligned} \phi^i &= g(r)e^{\mathrm{i}\omega t} \\ g(r) &= g_0\frac{\sin(\omega r+\delta)}{\omega r}. \end{aligned} \tag{3.183}$$

The phase shift of $\delta \sim M \neq 0$ is crucial in order to obtain Newtonian gravity. The time-component g_{00} has then approximately the form of a Schwarzschild-de Sitter metric with a constant shift

$$g_{00} \approx -\left(1 + 2U_0 - \frac{2GM}{r} - \frac{1}{3}\Lambda_{\text{eff}} r^2\right) \tag{3.184}$$

assuming

$$\frac{MG}{U_0} < r < L_\omega, \quad \text{where} \quad U_0 = -\frac{1}{2}g_0^2\omega^2 \quad \text{and} \quad \Lambda_{\text{eff}} = 2U_0\omega^2. \tag{3.185}$$

One finds an effective vacuum energy inside the gravity bag that can be considerably larger then the currently observed one. Moreover this solutions leads to a significant enhancement of the galactic rotation velocities at large distances which is actually observed but usually explained by large amounts of dark matter. It seems possible that in case of the harmonic branch far less dark matter might be necessary.

Concluding this section, one has to say that the studies of matter distributions are in their early phase and they are not conclusive yet[12]. More work is required to give a final answer to the question whether the dynamics of matter in our universe is predicted correctly by noncommutative emergent gravity.

[12]For instance, the radial component g_{rr} of the metric is not in agreement with the Schwarzschild-solution since it comes with an additional factor 1/3. This issue has not been resolved so far.

Chapter 4

Fermions coupled to noncommutative emergent gravity

In this chapter we turn to the central part of this work: The coupling of fermions to the matrix model and the evaluation of the induced one-loop effective action. We will observe that the coupling of fermions is non-standard due to the vanishing of the spin connection in matrix coordinates. This forbids us to take known results that can be found in the literature for the induced action at one-loop. Thus we have to evaluate explicitly the second Seeley-de Witt coefficient. The results of this chapter have been published in [52]. From now on we consider the Euclidean case for the sake of rigor.

4.1 The Dirac operator

To begin with, let us study the coupling of fermions to the matrix model action. We include fermions in our framework through the following action as suggested by the IKKT model Eq. (3.117)

$$S[\Psi] = (2\pi)^n \operatorname{Tr} \bar{\Psi}\gamma_a[X^a, \Psi], \tag{4.1}$$

where Ψ are Grassmann-valued matrices describing the fermions. It is worthwhile mentioning that fermions should be in the adjoint in this model, otherwise they cannot acquire a kinetic term. This does not rule out applicability in particle physics, see e.g. [79, 80]. In Euclidean spacetime we have for $\bar{\Psi}$

$$\bar{\Psi} = \Psi^\dagger. \tag{4.2}$$

Just like for scalars, also for fermions the only possibility to obtain a kinetic term is through commutators. The matrices γ_a generate the Clifford algebra in D dimensions

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}. \tag{4.3}$$

We split the action according to Sect. 3.1.3 and evaluate the semi-classical limit

$$\begin{aligned} S[\Psi] &= (2\pi)^n \operatorname{Tr} \bar{\Psi}\gamma_a[X^a, \Psi] \\ &= (2\pi)^n \operatorname{Tr} (\bar{\Psi}\gamma_\mu[X^\mu, \Psi] + \bar{\Psi}\gamma_i[\phi^i, \Psi]) \\ &\sim \int \mathrm{d}^{2n}x\, \rho(x)\bar{\Psi}\mathrm{i}(\gamma_\mu + \gamma_{3+i}\partial_\mu\phi^i)\theta^{\mu\nu}(x)\partial_\nu\Psi \\ &= \int \mathrm{d}^{2n}x\, \rho(x)\bar{\Psi}\mathrm{i}\widetilde{\gamma}_\mu\theta^{\mu\nu}\partial_\nu\Psi. \end{aligned} \tag{4.4}$$

Since we want to consider the embedding of a 3-brane in the ambient $\mathbb{R}^D$ we take $n = 2$. We have introduced the "tangential" Clifford algebra associated with the background metric $g_{\mu\nu}(x)$ on $\mathcal{M}_\theta$ whose elements are given by

$$\widetilde{\gamma}_\mu(x) = \left(\gamma_\mu + \gamma_{3+i}\partial_\mu\phi^i\right), \quad i = 2n, \ldots, D \tag{4.5}$$

and which satisfy

$$\{\widetilde{\gamma}_\mu, \widetilde{\gamma}_\nu\} = 2\left(\eta_{\mu\nu} + 2(\partial_\mu\phi^i)(\partial_\nu\phi^j)\delta_{ij}\right) = 2g_{\mu\nu}(x). \tag{4.6}$$

Notice that the $\widetilde{\gamma}(x)$ are functions of x and as such they are related to γ_μ via some vielbein which connects the Clifford representation $\widetilde{\gamma}^\mu(x)$ on curved space to the standard Dirac matrices γ^μ on Euclidean space,

$$\gamma^i = \begin{pmatrix} 0 & -\mathrm{i}\sigma^i \\ \mathrm{i}\sigma^i & 0 \end{pmatrix}, \quad \text{and} \quad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad i = 1, 2, 3 \tag{4.7}$$

with σ^i being the Pauli matrices.

The (matrix) Dirac operator is then given by

$$\begin{aligned} \not{D}\Psi &= \gamma_a\left[X^a, \Psi\right] \sim \mathrm{i}\left(\gamma_\mu + \gamma_{3+i}\left(\partial_\mu\phi^i\right)\right)\theta^{\mu\nu}(x)\partial_\nu\Psi \\ &= \mathrm{i}\,\widetilde{\gamma}_\mu\theta^{\mu\nu}\partial_\nu\Psi. \end{aligned} \tag{4.8}$$

This result for the Dirac operator in the semi-classical limit does not match the standard covariant derivative for spinors on curved spactime [72, 57] which reads as

$$\not{D}_{\mathrm{comm}}\Psi = \mathrm{i}\gamma^\alpha e^\mu_\alpha\left(\partial_\mu + \Sigma_{\beta\gamma}\omega_\mu^{\beta\,\gamma}\right)\Psi, \tag{4.9}$$

where

$$\omega_\mu^{\alpha\,\beta} = \frac{\mathrm{i}}{2}e^{\alpha\nu}\nabla_\mu e^\beta_\nu \tag{4.10}$$

is the usual spin connection, $\Sigma_{\alpha\beta} = \frac{\mathrm{i}}{4}[\gamma_\alpha, \gamma_\beta]$, and e^μ_α is the vielbein. While the explicit derivative term is the same, the spin connection vanishes in matrix coordinates x^μ in Eq. (4.8).

The spin connection determines how a spinor is rotated under parallel transport along a trajectory. However, along an open trajectory the spin connection can always be eliminated e.g. by a suitable gauge choice. Therefore in the point-particle limit, the trajectory of a fermion with the action Eq. (4.4) will follow properly the geodesics albeit with a different rotation of the spin.

As for the transport along a closed curve, it determines the holonomy in a gravitational field, and the non-standard spin connection in the above action strongly suggests that holonomies here will be different than in general relativity. The gravitational spin rotation for a free-falling fermion might provide a nice signature for or against the emergent gravity framework. This would need more elaboration.

A further remark is in order. In the case of extra dimensions the Poisson structure $\theta^{\mu\nu}$ does not play the sole rôle of a vielbein, rather it is part of a vielbein structure composed of the Poisson structure $\theta^{\mu\nu}$ and the derivatives of the embedding functions, $\partial_\mu \phi^i$.

The effective action for fermions at tree level is different from the effective action $\widetilde{G}_{\mu\nu}$ we had before. It is given by

$$\widetilde{G}^{\mu\nu}_{(\tau)}(x) = e^{-\tau}\theta^{\mu\alpha}\theta^{\nu\beta}g_{\alpha\beta}, \tag{4.11}$$

The difference is due to the new scaling factor $e^{-\tau}$ which is found to be

$$e^{-\tau} = |G_{\mu\nu}|^{1/6}|g_{\mu\nu}|^{-1/6}. \tag{4.12}$$

Compared to the previous scaling factor $e^{-\sigma}$ we have

$$e^{-\sigma} = -e^{\frac{3}{2}\tau}. \tag{4.13}$$

This can be seen by the following consideration.

$$\int \mathrm{d}^{2n}x\, \rho(x)\, \bar{\Psi}\, \mathrm{i}\, \widetilde{\gamma}_\mu \theta^{\mu\nu}\partial_\nu \Psi = \int \mathrm{d}^{2n}x\, \rho(x) e^{\tau/2}\, \bar{\Psi}\, \mathrm{i}\, e^{-\tau/2}\, \widetilde{\gamma}_\mu\, \theta^{\mu\nu}\partial_\nu \Psi. \tag{4.14}$$

The composed term

$$\Gamma^\nu \equiv e^{-\tau/2}\widetilde{\gamma}_\mu \theta^{\mu\nu} \tag{4.15}$$

defines the Clifford algebra associated to the effective metric $\widetilde{G}^{\mu\nu}_{(\tau)}(x)$,

$$\{\Gamma^\mu, \Gamma^\nu\} = e^{-\tau}\, \{\widetilde{\gamma}_\alpha, \widetilde{\gamma}_\beta\}\, \theta^{\alpha\mu}\theta^{\beta\nu} = 2\, e^{-\tau}\theta^{\alpha\mu}\theta^{\beta\nu}g_{\alpha\beta} = 2\, \widetilde{G}^{\mu\nu}_{(\tau)}(x). \tag{4.16}$$

In order to obtain a covariant action we demand

$$\rho(x)e^{\tau/2} \stackrel{!}{=} \sqrt{|\widetilde{G}_{(\tau)\mu\nu}|}, \tag{4.17}$$

which is fulfilled for the scaling factor of Eq. (4.12). The covariant action for fermions in the semi-classical limit is thus given by

$$S[\Psi] = \int \mathrm{d}^{2n}x \sqrt{|\widetilde{G}_{(\tau)\mu\nu}|}\, \bar{\Psi}\, \mathrm{i}\, \Gamma^\mu \partial_\mu \Psi. \tag{4.18}$$

4.2 Quantization

Due to the vanishing of the spin connection in matrix coordinates, a priori it is not clear how the induced one-loop effective action after integrating out the fermions will look like. Hence the central question of this work is whether the correct Einstein-Hilbert action will be induced despite the unusual form of the Dirac operator. In other words we ask whether the Dirac operator of Eq. (4.8) is an appropriate operator for one of the central ideas of this approach: The Einstein-Hilbert action emerges at the one-loop level. It is hence necessary to compute the induced action directly. This will be done in Sect. 4.3. Let us study first the quantization of the model.

Starting from the action

$$S[\Psi] = (2\pi)^n \operatorname{Tr} \bar{\Psi}\gamma_a[X^a, \Psi] \tag{4.19}$$

we want to study the quantization of this matrix model via a path integral over Ψ,

$$\begin{aligned} e^{-\Gamma_\Psi} &= \int \mathrm{d}\Psi \mathrm{d}\bar{\Psi} e^{-S[\Psi]} \\ &= \exp(\log\det(\not{D})) = \exp\left(\frac{1}{2}\log\det(\not{D})^2\right) \\ &= \exp\left(\frac{1}{2}\operatorname{Tr}\log(\not{D})^2\right). \end{aligned} \tag{4.20}$$

The effective action Γ_Ψ is then

$$\Gamma_\Psi = -\frac{1}{2}\operatorname{Tr}\log \not{D}^2. \tag{4.21}$$

The square of the Dirac operator takes the following form

$$\begin{aligned} \not{D}^2\Psi &= -\widetilde{\gamma}_\mu\widetilde{\gamma}_\rho\theta^{\mu\nu}\theta^{\rho\sigma}\partial_\nu\partial_\sigma\Psi - \widetilde{\gamma}_\mu\widetilde{\gamma}_\rho\theta^{\mu\nu}\left(\partial_\nu\theta^{\rho\sigma}\right)\left(\partial_\sigma\Psi\right) - \widetilde{\gamma}_\mu\left(\partial_\nu\widetilde{\gamma}_\rho\right)\theta^{\mu\nu}\theta^{\rho\sigma}\partial_\sigma\Psi \\ &= -G^{\mu\nu}\partial_\mu\partial_\nu\Psi - a^\mu\partial_\mu\Psi, \end{aligned} \tag{4.22}$$

where

$$G^{\mu\nu}(x) = \theta^{\mu\alpha}(x)\theta^{\nu\beta}(x)g_{\alpha\beta}(x), \tag{4.23}$$

and a^μ is the term linear in the partial derivatives

$$a^\mu = \widetilde{\gamma}_\sigma\widetilde{\gamma}_\rho\theta^{\sigma\nu}\left(\partial_\nu\theta^{\mu\sigma}\right) + \widetilde{\gamma}_\sigma\left(\partial_\nu\widetilde{\gamma}_\rho\right)\theta^{\sigma\nu}\theta^{\rho\mu}. \tag{4.24}$$

To proceed, we consider the quadratic form defined by $\not{D}^2$

$$\begin{aligned} S_{\text{square}} &:= (2\pi)^n \operatorname{Tr}\Psi^\dagger\not{D}^2\Psi \sim \int \mathrm{d}^{2n}x\, \rho(x)\Psi^\dagger\not{D}^2\Psi \\ &= \int \mathrm{d}^{2n}x\, \rho(x)\Psi^\dagger\left(-G^{\mu\nu}\partial_\mu\partial_\nu - a^\mu\partial_\mu\right)\Psi \\ &= \int \mathrm{d}^{2n}x\sqrt{|\widetilde{G}|}\Psi^\dagger\widetilde{\not{D}}^2\Psi, \end{aligned} \tag{4.25}$$

where we have already determined the appropriate scaling factor $e^{-\sigma}$ for the Dirac operator in oder to obtain the covariant form of the action S_{square} in the last step. The correct result for the scaling factor is

$$\begin{aligned} e^{-\sigma} &= \rho(x) = |G_{\mu\nu}|^{1/4}|g_{\mu\nu}|^{-1/4}, \\ \widetilde{G}^{\mu\nu} &= e^{-\sigma}G^{\mu\nu}. \end{aligned} \tag{4.26}$$

The rescaled Dirac operator $\widetilde{\not{D}}$ is then

$$\widetilde{\not{D}}^2 = -\widetilde{G}^{\mu\nu}\partial_\mu\partial_\nu - e^{-\sigma}a^\mu\partial_\nu. \tag{4.27}$$

Note that the scaling factor for the one-loop action $e^{-\sigma}$ is different from the scaling factor $e^{-\tau}$ of the covariant tree-level action Eq. (4.18),

$$e^{-\sigma} = e^{-\frac{3}{2}\tau}, \tag{4.28}$$

but equals the one for scalar fields, see Eq. (3.36) of Sect. 3.1.3. In order to compute the effective action we can use the following integral representation of the functional determinant

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}\left(\log\widetilde{\not{D}}^2\right) &= -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha}\left(e^{-\alpha\,\widetilde{\not{D}}^2}\right) \\ &\equiv -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha}\left(e^{-\alpha\,\widetilde{\not{D}}^2}\right)e^{-\frac{1}{\alpha\Lambda^2}}, \end{aligned} \tag{4.29}$$

where we have introduced a cutoff Λ^2 which regularizes the divergence of $\widetilde{\not{D}}^2$ for small α. Now we can apply the standard technique of the heat kernel expansion [63, 62]

$$\mathrm{Tr}e^{-\alpha\widetilde{\not{D}}^2} = \sum_{n\geq 0}\alpha^{\frac{n-4}{2}}\int_{\mathcal{M}}\mathrm{d}^4x\,\sqrt{|\widetilde{G}_{\mu\nu}|}\,a_n(x,\widetilde{\not{D}}^2) \tag{4.30}$$

to evaluate the integration over α. In the above equation $a_n(y,\widetilde{\not{D}}^2)$ are the Seeley-de Witt coefficients. For fermions the first two coefficients are given by [62]

$$\begin{aligned} a_0(x) &= \frac{1}{16\,\pi^2}\,\mathrm{tr}\mathbb{1}, \\ a_2(x) &= \frac{1}{16\,\pi^2}\,\mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\mathbb{1} + \mathcal{E}\right). \end{aligned} \tag{4.31}$$

Here tr denotes the trace over the spinorial matrices and $\mathcal{E}$ is given by

$$\mathcal{E} = -\widetilde{G}^{\mu\nu}\left(\partial_\mu\Omega_\nu + \Omega_\mu\Omega_\nu - \widetilde{\Gamma}^\rho_{\mu\nu}\Omega_\rho\right), \tag{4.32}$$

where

$$\Omega_\mu = \frac{1}{2}\widetilde{G}_{\mu\nu}(e^{-\sigma}a^\nu + \widetilde{G}^{\rho\sigma}\widetilde{\Gamma}^\nu_{\rho\sigma}). \tag{4.33}$$

Note that this expression is valid only in the matrix coordinates x^μ. These Seeley-de Witt coefficients give rise to the effective action

$$\Gamma_\Psi = \frac{1}{16\pi^2}\int d^{2n}x\sqrt{|\widetilde{G}|}\left(2\,\mathrm{tr}(\mathbb{1})\Lambda^{2n} + \mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\mathbb{1} + \mathcal{E}\right)\Lambda^2 + O(\log\Lambda)\right). \tag{4.34}$$

For the standard coupling of Dirac fermions to gravity on commutative spaces, one has [62]

$$\mathrm{tr}\mathcal{E}_{\mathrm{comm}} = -R, \tag{4.35}$$

so that the induced action usually gives the correct Einstein-Hilbert action. As mentioned before, this is the idea of emergent gravity observed first by Sakharov [60]. In our case $\mathrm{tr}\mathcal{E}$ is modified due to the non-standard spin connection. Therefore we cannot use the standard results, and the geometrical meaning of Eq. (4.33) is unclear since this expression is not covariant and valid only in matrix coordinates. The purpose of the next section is to evaluate the quantity $\mathrm{tr}\mathcal{E}$ and see whether it gives indeed the Ricci scalar $R[\widetilde{G}]$ in order to obtain the induced Einstein-Hilbert action. We will show that $\mathrm{tr}\mathcal{E}$ contains as expected the appropriate curvature scalar, plus three additional terms. This will be discussed in detail in the following sections.

4.3 Evaluation of $\mathrm{tr}\mathcal{E}$

We will now determine explicitly the second Seeley-de Witt coefficient for the squared Dirac operator Eq. (4.22). In order to do so we compute the Ricci scalar as well as the quantity $\mathrm{tr}\mathcal{E}$ explicitly in terms of the Poisson structure, and then compare those two. First we have to compute the following expression

$$\mathrm{tr}\mathcal{E} = -\mathrm{tr}\left\{\widetilde{G}^{\mu\nu}\Omega_\mu\Omega_\nu + \widetilde{G}^{\mu\nu}\partial_\mu\Omega_\nu - \widetilde{\Gamma}^\rho\Omega_\rho\right\}. \tag{4.36}$$

Since Ω_μ is given by

$$\Omega_\mu = \frac{1}{2}\widetilde{G}_{\mu\nu}\left(e^{-\sigma}a^\nu + \widetilde{G}^{\rho\sigma}\widetilde{\Gamma}^\nu_{\rho\sigma}\right) \tag{4.37}$$

we find

$$\mathrm{tr}\mathcal{E} = -\mathrm{tr}\left(\frac{1}{4}\widetilde{G}_{\mu\nu}\widetilde{a}^\mu\widetilde{a}^\nu - \frac{1}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^\mu\widetilde{\Gamma}^\nu + \frac{1}{2}\widetilde{G}^{\mu\nu}\partial_\mu\big(\widetilde{G}_{\nu\rho}\widetilde{a}^\rho + \widetilde{G}_{\nu\rho}\widetilde{\Gamma}^\rho\big)\right). \tag{4.38}$$

The explicit evaluation of tr$\mathcal{E}$ is rather lengthy and technical. It is given in Appendix A. The result stated in Eq. (A.19) is

$$
\begin{aligned}
\mathrm{tr}\mathcal{E} &= -\mathrm{tr}\left(\frac{1}{4}\widetilde{G}_{\mu\nu}\widetilde{a}^{\mu}\widetilde{a}^{\nu} - \frac{1}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^{\mu}\widetilde{\Gamma}^{\nu}\right)\\
&= -e^{-\sigma}\frac{k}{4}\Big\{ - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\\
&\quad + G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\sigma g_{\alpha\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\delta g_{\alpha\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta}\\
&\quad - G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\alpha g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\sigma g_{\beta\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta}\\
&\quad - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\delta g_{\beta\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\rho\sigma}\theta^{\mu\beta}(\partial_\beta g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta}\\
&\quad - \frac{1}{2}\big(G^{\mu\nu}(g\partial_\mu\partial_\nu g^{-1}) - 2G^{\rho\sigma}g^{\delta\beta}\partial_\rho\partial_\beta g_{\sigma\delta} + g^{\mu\nu}G^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}\big)\\
&\quad + \frac{1}{2}\theta^{\rho\alpha}(\partial_\rho g_{\gamma\beta})\theta^{\sigma\gamma}(\partial_\sigma g_{\alpha\delta})g^{\delta\beta} + \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\mu g_{\rho\alpha})(\partial_\nu g_{\sigma\beta})g^{\alpha\beta}\\
&\quad - \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\alpha g_{\mu\rho})(\partial_\beta g_{\nu\sigma})g^{\alpha\beta}\Big\}\\
&\quad + \frac{k}{4}g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\big(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\rho\partial_\rho\phi^j\big)\delta_{ij} + \frac{k}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^{\mu}\widetilde{\Gamma}^{\nu},
\end{aligned}
\tag{4.39}
$$

where

$$k = \mathrm{rank}(\gamma). \tag{4.40}$$

k is the rank of the representation of the D-dimensional Clifford algebra, depending on the number of extra dimensions.

For the sake of simplicity and manageability we will sometimes use the equations of motions, Eq. (3.45) and (3.52) and work with on-shell geometries. Then the contracted Christoffel symbols vanish [50],

$$
\begin{aligned}
\widetilde{\Gamma}^{\mu} &= -\partial_\rho\widetilde{G}^{\rho\mu} - \frac{1}{2}\widetilde{G}^{\mu\nu}(\widetilde{G}^{\rho\sigma}\partial_\nu\widetilde{G}^{-1}_{\rho\sigma})\\
&= e^{-\sigma}\left(-(\partial_\rho\theta^{\mu\beta})\theta^{\rho\alpha}g_{\alpha\beta} - \theta^{\mu\beta}\theta^{\rho\alpha}(\partial_\rho g_{\alpha\beta})\right)\\
&\overset{\text{e.o.m.}}{=} 0.
\end{aligned}
\tag{4.41}
$$

Due to Eq.(4.41) also the harmonic embedding condition simplifies as

$$
\begin{aligned}
\Delta_{\widetilde{G}}\phi &= \left(\widetilde{G}^{\mu\nu}\partial_\mu\partial_\nu - \widetilde{\Gamma}^{\mu}\partial_\mu\right)\phi\\
&= \widetilde{G}^{\mu\nu}\partial_\mu\partial_\nu\phi\\
&= 0.
\end{aligned}
\tag{4.42}
$$

These handy features simplify our calculations a lot since tr$\mathcal{E}$ is then determined by a single term,

$$\mathrm{tr}\mathcal{E} = -\frac{e^{-\sigma}}{4}\mathrm{tr}\left(G_{\mu\nu}a^{\mu}a^{\nu}\right). \tag{4.43}$$

In principle one could now go on and compute the Ricci scalar in terms of the Poisson tensor $\theta^{\mu\nu}$ and compare the two quantities. This strategy will be pursued in Chapter 5 for the special case of $D = 4$. However, it turns out that this procedure is too complicated and does not seem to be feasible in the case of extra dimensions. Hence we simplify our computations by going to *normal coordinates*, i.e. coordinates where first order derivatives in the embedding scalar fields $\partial_\mu\phi(x)$ vanish. But in order to be allowed to perform this step, we should first show that the quantitiy $\mathrm{tr}\mathcal{E}$ is a covariant expression. This is done in Appendix B, and we only quote the result here. For on-shell geometries which satisfy Eq. (3.45) and (3.52), we find

$$\begin{aligned}\mathrm{tr}\mathcal{E} = -\frac{e^{-\sigma}}{4}\, k \Big(& G^{\mu\nu}(\nabla_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\nabla_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ & + G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\Big) - \frac{k}{4}\widetilde{G}^{\mu\nu}R_{\mu\nu}[g],\end{aligned} \tag{4.44}$$

see also Eq. (B.22) of Appendix B. In the special case of identical background and effective metric $\widetilde{G}_{\mu\nu} = g_{\mu\nu}$ the use of on-shell geometries is not necessary. Then we have

$$\begin{aligned}\mathrm{tr}\mathcal{E} &= -\frac{e^{-\sigma}}{4}\mathrm{tr}\, G_{\mu\nu}a^\mu a^\nu + \frac{e^{-\sigma}}{4}\mathrm{tr}\, g_{\mu\nu}\Gamma^\mu\Gamma^\nu \\ &= -\frac{e^{\sigma}}{4}\, k\, g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} - \frac{k}{4}R[g] + \frac{k}{4}(\Delta_g x^a)(\Delta_g x^b)\eta_{ab},\end{aligned} \tag{4.45}$$

as stated in Eq. (B.44). This expression has a clear geometrical meaning (taking into account extrinsic geometry in the last term) and is thus covariant. For on-shell geometries, the last term vanishes so that Eq. (4.45) agrees with Eq. (4.44) using the Bianchi identity (B.37).

A short remark regarding notation. We have to distinguish between the *effective metric* $\widetilde{G}_{\mu\nu}$ and the *background metric* $g_{\mu\nu}$. Covariant derivatives and Christoffel symbols with respect to the effective metric $\widetilde{G}_{\mu\nu}$ in NC emergent gravity are usually denoted as $\widetilde{\nabla}_\mu$ and $\widetilde{\Gamma}^\mu_{\rho\sigma}$, respectively. Covariant derivatives and Christoffel symbols with respect to the background metric $g_{\mu\nu}$ as they appear in Eq. (4.44) and (4.45) are written as ∇_μ and $\Gamma^\mu_{\rho\sigma}$.

4.4 Normal coordinates

In order to be able to compare the Ricci scalar $R[\widetilde{G}]$ to $\mathrm{tr}\mathcal{E}$ it is necessary to simplify the corresponding expressions for these two quantities which can be evaluated in terms of the Poisson structure and the embedding fields. Choosing an appropriate coordinate system is of great advantage. It turns out that a normal coordinate system simplifies the technical issues to a sufficiently high degree. This procedure will be described in the following. Due to technical complications we focus on two special cases:

1. "On-shell geometries" as determined by the semi-classical equations of motion (3.45) and (3.52) of the matrix model.

2. The class of geometries where the effective metric $\widetilde{G}^{\mu\nu}$ coincides with the induced metric $g^{\mu\nu}$. This class seems to be general enough for a large class of physical situations, see [53, 54] and Sect. 3.5.

4.4.1 tr$\mathcal{E}$ in normal coordinates for on-shell geometries

Since the matrix model action is invariant under $SO(D)$ respectively $SO(1, D-1)$ rotations as well as translations, one can choose for any given point $p \in \mathcal{M}_\theta$ adapted coordinates such that the brane is tangential to the plane spanned by the first $2n$ components. Then we have at this point

$$\partial_\mu \phi^i|_p = 0, \tag{4.46}$$
$$\partial_\mu g_{\rho\sigma}|_p = 0. \tag{4.47}$$

We denote such coordinates as "normal embedding coordinates" or simply "normal coordinates". They are still matrix coordinates $x^\mu \sim X^\mu$ and thus the e.o.m. (3.45) and (3.52) still hold. We can now take our result of Eq. (4.39) for tr$\mathcal{E}$ and write it in normal coordinates by simply omitting all terms with first partial derivatives of the background metric $g_{\mu\nu}$. Since the covariance of tr$\mathcal{E}$ of Eq. (4.44) for general $\widetilde{G}$ is only established by making use of the e.o.m. we have to work with on-shell geometries. tr$\mathcal{E}$ in normal coordinates is thus given by

$$\begin{aligned}
\mathrm{tr}\mathcal{E} &= -\frac{e^{-\sigma}}{4}\mathrm{tr}\left\{G_{\mu\nu}a^\mu a^\nu\right\} \\
&= e^{-\sigma}k\left\{\frac{1}{4}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - \frac{1}{4}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\right. \\
&\quad \left. + \frac{1}{8}\big(G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}) - 2G^{\rho\sigma}g^{\delta\beta}\partial_\rho\partial_\beta g_{\sigma\delta} + g^{\mu\nu}G^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}\big)\right\} \\
&\quad + \frac{k}{4}g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\rho(\partial_\rho\phi^j)\delta_{ij}\right) + \frac{k}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^\mu\widetilde{\Gamma}^\nu.
\end{aligned} \tag{4.48}$$

This expression can be simplified further by making use of the relation Eq. (B.2) of Appendix A

$$(\partial_\lambda\phi^i)(\partial_\mu\partial_\nu\phi^j)\delta_{ij} = \frac{1}{2}\big(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\big). \tag{4.49}$$

Differentiating once more gives

$$(\partial_\rho\partial_\sigma\phi^i)(\partial_\mu\partial_\nu\phi^j)\delta_{ij} \stackrel{\text{nc}}{=} \frac{1}{2}\left(\partial_\rho\partial_\mu g_{\nu\sigma} + \partial_\rho\partial_\nu g_{\mu\sigma} - \partial_\rho\partial_\sigma g_{\mu\nu}\right), \tag{4.50}$$

where the superscript "nc" stands for normal coordinates. In normal coordinates we have

$$\begin{aligned} g^{\rho\sigma}G^{\mu\nu}(\partial_\rho\partial_\mu g_{\nu\sigma}) &= g^{\rho\sigma}(\partial_\rho\partial_\sigma\phi^i)G^{\mu\nu}(\partial_\mu\partial_\nu\phi^j)\delta_{ij} + \frac{1}{2}g^{\mu\nu}G^{\rho\sigma}(\partial_\rho\partial_\sigma g_{\mu\nu}) \\ &= e^\sigma g^{\rho\sigma}(\partial_\rho\partial_\sigma\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\mu(\partial_\mu\phi^j)\right)\delta_{ij} + \frac{1}{2}g^{\mu\nu}G^{\rho\sigma}(\partial_\rho\partial_\sigma g_{\mu\nu}). \end{aligned} \tag{4.51}$$

Our final result for $\mathrm{tr}\mathcal{E}$ in normal coordinates is then

$$\begin{aligned} \mathrm{tr}\mathcal{E} &= e^{-\sigma}\frac{k}{4}\Big\{G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\ &\quad + \frac{1}{8}G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma})\Big\} + \frac{k}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^\mu\widetilde{\Gamma}^\nu \\ &\overset{\text{e.o.m.}}{=} e^{-\sigma}\frac{k}{4}\Big\{G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\ &\quad + \frac{1}{8}G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma})\Big\}, \end{aligned} \tag{4.52}$$

where we finally have exploited the e.o.m. in the last step.

4.4.2 $\mathrm{tr}\mathcal{E}$ in normal coordinates for $\widetilde{G} = g$

Since it was shown in the last section that $\mathrm{tr}\mathcal{E}$ is a covariant expression for $\widetilde{G} = g$ even for off-shell geometries, we will not use the e.o.m. here. The term $\frac{k}{4}g_{\mu\nu}\Gamma^\mu\Gamma^\nu$ vanishes now due to the normal coordinate system. Since $\theta^{-1}_{\mu\nu}$ fulfills the Jacobi identity the following equation

$$2(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta} = (\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta} \tag{4.53}$$

holds for $\widetilde{G} = g$, see also Appendix B. This simplifies $\mathrm{tr}\mathcal{E}$ to

$$\mathrm{tr}\mathcal{E} = e^\sigma\frac{k}{4}g^{\mu\nu}g^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} + \frac{k}{8}g^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}). \tag{4.54}$$

4.4.3 The Ricci scalar $R[\widetilde{G}]$ in normal coordinates

Let us now study the Ricci scalar $R[\widetilde{G}]$ in normal coordinates. The curvature tensor and the Ricci scalar are given as usual by

$$\begin{aligned} R_{\mu\nu\rho}{}^{\sigma}[\widetilde{G}] &= \partial_\nu\widetilde{\Gamma}^\sigma_{\mu\rho} - \partial_\mu\widetilde{\Gamma}^\sigma_{\nu\rho} + \widetilde{\Gamma}^\lambda_{\mu\rho}\widetilde{\Gamma}^\sigma_{\lambda\nu} - \widetilde{\Gamma}^\lambda_{\nu\rho}\widetilde{\Gamma}^\sigma_{\lambda\mu}, \\ R[\widetilde{G}] &= \widetilde{G}^{\mu\rho}R_{\mu\nu\rho}{}^{\nu}. \end{aligned} \tag{4.55}$$

In terms of the metric (now with respect to the effective metric $\widetilde{G}$) and its derivatives the Ricci scalar is given by

$$
\begin{aligned}
R[\widetilde{G}] = &-\widetilde{G}_{\mu\nu}(\partial_\rho \widetilde{G}^{\rho\mu})(\partial_\sigma \widetilde{G}^{\sigma\nu}) + \widetilde{G}^{\mu\nu}\widetilde{G}^{\rho\sigma}(\partial_\mu\partial_\rho \widetilde{G}_{\nu\sigma}) \\
&- \widetilde{G}^{\mu\nu}\widetilde{G}^{\rho\sigma}\partial_\rho\partial_\sigma \widetilde{G}_{\mu\nu} - (\partial_\rho \widetilde{G}^{\rho\sigma})(\widetilde{G}^{\mu\nu}\partial_\sigma \widetilde{G}_{\mu\nu}) \\
&- \frac{3}{4}\widetilde{G}^{\mu\nu}(\partial_\mu \widetilde{G}^{\rho\sigma})(\partial_\nu \widetilde{G}_{\rho\sigma}) + \frac{1}{2}\widetilde{G}^{\rho\sigma}(\partial_\sigma \widetilde{G}^{\mu\nu})(\partial_\nu \widetilde{G}_{\mu\rho}) \\
&- \frac{1}{4}\widetilde{G}^{\mu\nu}(\widetilde{G}^{\rho\sigma}\partial_\mu \widetilde{G}_{\rho\sigma})(\widetilde{G}^{\kappa\lambda}\partial_\nu \widetilde{G}_{\kappa\lambda}).
\end{aligned}
\tag{4.56}
$$

$R[\widetilde{G}]$ in normal coordinates for on-shell geometries. See Appendix C for the evaluation of the Ricci scalar in normal coordinates. The result is found to be

$$
\begin{aligned}
R[\widetilde{G}] \overset{\text{nc}}{=} e^{-\sigma}\Big\{ &\frac{1}{2}(\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})\eta_{\alpha\beta} + \frac{1}{2}(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})\eta_{\alpha\beta} \\
&+ \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})\eta^{\alpha\beta} - \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})\eta^{\alpha\beta} \\
&- \frac{1}{2}G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma})\Big\}.
\end{aligned}
\tag{4.57}
$$

$R[\widetilde{G}]$ in normal coordinates for $\widetilde{G} = g$. If the background metric equals the effective metric, the Ricci scalar in normal coordinates is due to Eq. (4.51)

$$
\begin{aligned}
R[g] &= g^{\mu\nu}g^{\rho\sigma}(\partial_\mu\partial_\rho g_{\nu\sigma}) - g^{\mu\nu}g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\mu\nu} \\
&= -\frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\partial_\rho\partial_\sigma g_{\mu\nu} + (\Delta_g\phi^i)(\Delta_g\phi^j)\delta_{ij}.
\end{aligned}
\tag{4.58}
$$

Hence, in that special case we have

$$
\begin{aligned}
(\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + (\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} &= -e^{2\sigma}g^{\mu\nu}g^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
&\quad + e^{2\sigma}g^{\mu\nu}g^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&= e^{2\sigma}g^{\mu\nu}g^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}.
\end{aligned}
\tag{4.59}
$$

4.4.4 A comparison of tr$\mathcal{E}$ & $R[g]$

Let us finally compare our results for tr$\mathcal{E}$ and the Ricci scalar $R[\widetilde{G}]$. As before we separate the two cases of interest.

A comparison for on-shell quantities tr$\mathcal{E}$ and $R[\widetilde{G}]$. In normal coordinates we find the following relation between the Ricci scalar and tr$\mathcal{E}$,

$$
\begin{aligned}
\mathrm{tr}\mathcal{E} = &-\frac{k}{2}R[\widetilde{G}] - \frac{k}{8}G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}) \\
&+ \frac{k}{4}e^{-\sigma}(\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + \frac{k}{4}e^{-\sigma}(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta}.
\end{aligned}
\tag{4.60}
$$

We can see that tr$\mathcal{E}$ contains the Ricci scalar which was expected. However, there are also additional contributions which we will now write in terms of covariant expressions in order to clarify their geometrical meaning. In order to do so, we notice that

$$\begin{aligned}\theta^{\mu\alpha}(\nabla_\mu\theta^{\nu\beta})g_{\alpha\beta} &= \theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})g_\alpha + \theta^{\mu\alpha}\Gamma^\nu_{\mu\rho}\theta^{\rho\beta}g_{\alpha\beta} + \theta^{\mu\alpha}\Gamma^\beta_{\mu\rho}\theta^{\nu\rho}g_{\alpha\beta} \\ &= \theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})g_\alpha + \Gamma^\nu_{\rho\sigma}G^{\rho\sigma} + \theta^{\mu\alpha}\theta^{\nu\rho}(\partial_\mu g_{\alpha\rho}) \\ &= \Gamma^\nu_{\rho\sigma}G^{\rho\sigma} \\ &= G^{\rho\sigma}(\partial_\rho g_{\lambda\sigma})g^{\lambda\nu} - \frac{1}{2}G^{\rho\sigma}g^{\lambda\nu}(\partial_\lambda g_{\rho\sigma}) \\ &= 0,\end{aligned} \tag{4.61}$$

using the e.o.m and Eq. (4.50). A consequence of the above relation is then

$$(\nabla_\mu\theta^{\nu\alpha})(\nabla_\nu\theta^{\mu\beta})g_{\alpha\beta} = -\theta^{\mu\alpha}(\nabla_\nu\nabla_\mu\theta^{\nu\beta})g_{\alpha\beta}. \tag{4.62}$$

In normal coordinates this expression is

$$(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} = -\theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} - \theta^{\mu\alpha}\theta^{\nu\beta}\partial_\mu\partial_\nu g_{\alpha\beta}, \tag{4.63}$$

which can also be derived from the equation of motion. Now remember that

$$\begin{aligned}\theta^{\mu\alpha}(\nabla_\nu\nabla_\mu\theta^{\nu\beta})g_{\alpha\beta} &= -\theta^{\mu\alpha}(\nabla_\mu\nabla_\nu\theta^{\nu\beta})g_{\alpha\beta} + R^\mu_{\lambda\mu\nu}G^{\lambda\nu} + R^\beta_{\lambda\mu\nu}\theta^{\mu\lambda}\theta^{\nu\alpha}g_{\alpha\beta} \\ &= -\theta^{\mu\alpha}(\nabla_\mu\nabla_\nu\theta^{\nu\beta})g_{\alpha\beta} + G^{\mu\nu}R[g]_{\mu\nu} - \frac{1}{2}R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}.\end{aligned} \tag{4.64}$$

Next consider the Ricci tensor in normal coordinates,

$$\begin{aligned}R_{\mu\nu} &\overset{\text{nc}}{=} \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\mu\Gamma^\rho_{\rho\nu} \\ &= \frac{1}{2}g^{\rho\lambda}\Big(2\partial_\mu\partial_\rho g_{\lambda\nu} + \partial_\nu\partial_\rho g_{\mu\lambda} - \partial_\rho\partial_\lambda g_{\mu\nu} + \partial_\mu\partial_\nu g_{\rho\lambda} - \partial_\mu\partial_\lambda g_{\nu\rho}\Big).\end{aligned} \tag{4.65}$$

Due to Eq.(4.51) we find then

$$G^{\mu\nu}R[g]_{\mu\nu} \overset{\text{nc}}{=} -\frac{1}{2}G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}). \tag{4.66}$$

Using also

$$\theta^{\mu\alpha}(\nabla_\mu\nabla_\nu\theta^{\nu\beta})g_{\alpha\beta} = G^{\mu\nu}\partial_\mu\partial_\nu\sigma = G^{\mu\nu}\nabla_\mu\nabla_\nu\sigma \tag{4.67}$$

as well as

$$(\nabla_\mu\theta^{\mu\alpha})(\nabla_\nu\theta^{\nu\beta})g_{\alpha\beta} \overset{\text{nc}}{=} (\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} = G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) = G^{\mu\nu}(\nabla_\mu\sigma)(\nabla_\nu\sigma) \tag{4.68}$$

we obtain the following covariant form of tr$\mathcal{E}$,

$$\mathrm{tr}\mathcal{E} = -\frac{k}{2}R[\widetilde{G}] + \frac{k}{4}\widetilde{G}^{\mu\nu}(\nabla_\mu\sigma)(\nabla_\nu\sigma) + \frac{k}{4}\widetilde{G}^{\mu\nu}\nabla_\mu\nabla_\nu\sigma + \frac{k}{8}\,e^{-\sigma}\,R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma} \tag{4.69}$$

or

$$\mathrm{tr}\mathcal{E} = -\frac{k}{2}R[\widetilde{G}] + \frac{k}{4}e^{-\sigma}\Delta_{\tilde{G}}e^{\sigma} + \frac{k}{8}\,e^{-\sigma}\,R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}. \tag{4.70}$$

Ultimately, we yield for on-shell geometries the following final result for the one-loop effective action:

$$\begin{aligned}\Gamma_\Psi = \frac{k}{16\pi^2}\int \mathrm{d}^{2n}x\sqrt{|\widetilde{G}|}\Big(2\Lambda^{2n} + \big(-\frac{1}{3}R[\widetilde{G}] + \frac{e^{-\sigma}}{4}\Delta_{\tilde{G}}e^{\sigma}\\ + \frac{e^{-\sigma}}{8}R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}\big)\Lambda^2 + O(\log\Lambda)\Big).\end{aligned} \tag{4.71}$$

A comparison of tr$\mathcal{E}$ and R for $\widetilde{G} = g$. Let us now study again the special case $\widetilde{G} = g$ without demanding on-shell conditions. Then tr$\mathcal{E}$ was given by

$$\begin{aligned}\mathrm{tr}\mathcal{E} &\overset{\mathrm{nc}}{=} \frac{k}{4}e^{-\sigma}\,\big((\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + (\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta}\big) - \frac{k}{4}R[g]\\ &\quad + \frac{k}{4}(\Delta_g\phi^i)(\Delta_g\phi^j)\delta_{ij}.\end{aligned} \tag{4.72}$$

Since

$$\begin{aligned}\theta^{\mu\alpha}(\nabla_\mu\theta^{\nu\beta})g_{\alpha\beta} &= \theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})g_{\alpha\beta} + G^{\mu\rho}\Gamma^{\nu}_{\mu\rho} + \theta^{\mu\alpha}\theta^{\nu\beta}(\partial_\nu g_{\alpha\beta})\\ &\overset{\widetilde{G}=g}{=} -e^{\sigma}\Gamma^{\nu} + e^{\sigma}\Gamma^{\nu}\\ &= 0\end{aligned} \tag{4.73}$$

the relation

$$(\nabla_\mu\theta^{\nu\alpha})(\nabla_\nu\theta^{\mu\beta})g_{\alpha\beta} = -\theta^{\mu\alpha}(\nabla_\nu\nabla_\mu\theta^{\nu\beta})g_{\alpha\beta} \tag{4.74}$$

is true also for off-shell geometries in the case of $\widetilde{G} = g$. With the help of Eq. (4.51) we see that

$$\begin{aligned}\theta^{\mu\alpha}(\nabla_\nu\nabla_\mu\theta^{\nu\beta})g_{\alpha\beta} &\overset{\mathrm{nc}}{=} \theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + \theta^{\mu\alpha}\theta^{\nu\beta}\partial_\mu\partial_\nu g_{\alpha\beta} + (\Delta_g\phi^i)(\Delta_g\phi^j)\delta_{ij}\\ &\overset{\mathrm{nc}}{=} -(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta}.\end{aligned} \tag{4.75}$$

Using Eq. (B.38) of Appendix B we have

$$\theta^{\mu\alpha}(\nabla_\mu\nabla_\nu\theta^{\nu\beta})g_{\alpha\beta} \overset{\mathrm{nc}}{=} \theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + \frac{e^{\sigma}}{2}g^{\mu\nu}g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma} \overset{\mathrm{nc}}{=} e^{\sigma}g^{\mu\nu}\partial_\mu\partial_\nu\sigma, \tag{4.76}$$

giving

$$\mathrm{tr}\mathcal{E} = -\frac{k}{2}R[g] + \frac{k}{4}e^{-\sigma}g^{\mu\nu}\nabla_\mu\nabla_\nu e^{\sigma} + \frac{k}{8}e^{-\sigma}R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma} + \frac{k}{4}(\Delta_g x^a)(\Delta_g x^b)\eta_{ab}. \tag{4.77}$$

The total one-loop effective action for $\widetilde{G} = g$ is thus

$$\begin{aligned}\Gamma_\Psi = \frac{k}{16\pi^2}\int \mathrm{d}^{2n}x\sqrt{|g|}\Big(2\Lambda^{2n} + \big(-\frac{1}{3}R[g] + \frac{1}{4}e^{-\sigma}\Delta_g e^{\sigma}\\ + \frac{1}{8}e^{-\sigma}R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma} + \frac{1}{4}(\Delta_g x^a)(\Delta_g x^b)\eta_{ab}\big)\Lambda^2 + O(\log\Lambda)\Big),\end{aligned} \tag{4.78}$$

without requiring the equations of motion. This agrees with Eq. (4.71) plus an additional term depending on the extrinsic geometry.

4.5 Discussion

In this chapter we have evaluated the one-loop effective action for fermions coupled to noncommutative emergent gravity. We were focusing on two special cases due to technical reasons: On-shell results for arbitrary effective metric $\widetilde{G}^{\mu\nu} \neq g^{\mu\nu}$ and off-shell results for the class of self-dual metrics where $\widetilde{G}_{\mu\nu} = g_{\mu\nu}$. The one-loop effective action for on-shell geometries was evaluated to be

$$\begin{aligned}\Gamma_\Psi = \frac{k}{16\pi^2}\int \mathrm{d}^{2n}x\,\sqrt{|\widetilde{G}|}\Bigg(2\Lambda^{2n} + \Big(-\frac{1}{3}R[\widetilde{G}] + \frac{e^{-\sigma}}{4}\,\Delta_{\widetilde{G}}e^{\sigma}\\ + \frac{e^{-\sigma}}{8}R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}\Big)\Lambda^2 + O(\log\Lambda)\Bigg).\end{aligned} \tag{4.79}$$

The one-loop effective action for $\widetilde{G}_{\mu\nu} = g_{\mu\nu}$ as stated above in Eq. (4.78) is similar. However, it contains an additional term $\frac{1}{4}(\Delta_g\, x^a)(\Delta_g\, x^b)\eta_{ab}$ which vanishes in the case of on-shell conditions.

Thus we have shown that the Einstein-Hilbert action is indeed induced albeit with a different coefficient. Moreover we have found two or three additional terms depeding on the two cases. The term

$$e^{-\sigma}\,\Delta_{\widetilde{G}}e^{-\sigma} \tag{4.80}$$

comes from the scalar density and it can be regarded as a reminiscent of a dilaton-like term. The term

$$e^{-\sigma}\,R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma} \tag{4.81}$$

is in some sense undesirable due to the explicit coupling of the Poisson tensor $\theta^{\mu\nu}$. This is the first term found in noncommutative emergent gravity where this explicit coupling happens at the semi-classical level. The problematic nature of this term lies in the fact that it will break Lorentz invariance. While this term vanishes in geometries with flat background metric $g_{\mu\nu}(x)$, in general it will not be irrelevant.

One can think of two solutions which might avoid relevent effects of Lorentz breaking. These solutions are related to the two "branches" of solutions given in Sect. 3.5.4.

- So far it is not clear which term in the induced action will be dominant. One possible scenario is that the term describing the brane tension
$$\int \mathrm{d}^{2n}x\, \sqrt{|\widetilde{G}|}\Lambda_{\text{vac}}^{2n} \tag{4.82}$$
is dominant and hence gravity is governed by harmonic embeddings. In that case both the Einstein-Hilbert action as well as all the other terms including the term $e^{-\sigma}\, R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$ will describe corrections which will have to be small enough such that the Lorentz breaking effects have not been observed so far. For this to be realized, the cutoff $\Lambda = \Lambda_{\text{EH}}$ as given by the breaking of $\mathcal{N} = 4$ supersymmetry is much smaller than the Planck scale, as shown in [54]. This solution corresponds to the "harmonic branch" which is preferable in the context of the cosmological constant problem as discussed in Sect. 3.5.3.

- Secondly, the "Einstein branch" could be realized. Then the scale of $\mathcal{N} = 4$ supersymmetry breaking should be identified with the Planck scale as
$$\Lambda = \Lambda_{\text{EH}} \approx \Lambda_{\text{Planck}}. \tag{4.83}$$
Since we know that there will be no induced (gravitational) action due to a precise cancellation,
$$\Gamma_A = -2\Gamma_\Psi - 6\Gamma_\Phi, \tag{4.84}$$
above the scale of $\mathcal{N} = 4$ supersymmetry breaking we conclude that also the problematic term has to cancel. Remember that the result for the induced action is the same whether it is obtained from the "gauge theory point of view" or the geometrical point of view. That contribution is expected to be cancelled due to the would-be topological term in Eq. (3.95) with opposite sign. It is not clear what exactely happens below the scale Λ. Especially, how the specific breaking mechanism of supersymmetry works is not known. It might be possible that below the scale Λ and above some lower scale - denoted as Λ_1 - the unbroken gauge fields and fermions are matched such that $n_A = 2n_\Psi$. If this were the case the unwanted term Eq. (4.81) would not be induced. The gravitational action induced due to integrating out all fields between Λ_1 and Λ is equivalent to the contributions due to scalars to an intermediate induced action
$$\Gamma_{\text{grav}} = \frac{n_\Phi}{16\pi^2}\int \mathrm{d}^{2n}x\, \Big(2\Lambda^{2n} + \frac{1}{6}R[\widetilde{G}]\Lambda^2 + O(\log\Lambda)\Big), \tag{4.85}$$
where n_Φ is the effective number of "missing" scalar fields from the $\mathcal{N} = 4$ spectrum.

Finally, it could also be that there is no way to get rid off this term. In that case its physical meaning has to be studied. It would certainly be worthwhile to investigate the consequences of this new term which has not played a rôle in noncommutative theories so far. Despite the unwelcomed feature of Lorentz breaking there still might be interesting physics included in this expression.

Chapter 5

Fermions and UV/IR mixing

In this chapter we come back to the duality between noncommutative gauge theory and gravity. We will interpret UV/IR mixing for fermions in noncommutative gauge theory as a gravitational effect. This part of the work is done in spirit of Sect. 3.3.2. We will also explain why some UV/IR mixing remains even in supersymmetric theories, except in the case $\mathcal{N} = 4$. For a reasonable comparison of these two interpretations we take the number of spacetime dimensions equal to four in the following. We evaluate once more $\mathrm{tr}\mathcal{E}$ and R which is much simpler in this case. This provides a nontrivial check that our geometrical results are correct because they can be compared directly to the known results from gauge theory. The results stated in this chapter have been published in [49].

5.1 Fermions in four dimensions

In this section we basically repeat the considerations of the previous chapter. We do this in order to clearly point out the differences with respect to the case of extra dimensions. The matrix model action becomes in the semi-classical limit

$$S[\Psi] = (2\pi)^2 \,\mathrm{Tr}\, \bar{\Psi}\gamma_\mu[X^\mu, \Psi] \sim \mathrm{d}^4x\, \rho(x)\, \bar{\Psi}\, \mathrm{i}\, \gamma_\mu \theta^{\mu\nu}\partial_\nu \Psi, \tag{5.1}$$

where γ_μ are elements of the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \tag{5.2}$$

where $g_{\mu\nu} = \delta_{\mu\nu}$[1] is now constant. In the four-dimensional case

$$e^\mu_\beta(x) := \theta^{\mu\alpha}\eta_{\alpha\beta} \tag{5.3}$$

plays the rôle of a vielbein defined in matrix coordinates. For $D = 4$ we do not need an embedding hence the matrix Dirac operator is given by

$$\not{D} = \mathrm{i}\gamma_\mu\theta^{\mu\nu}\partial_\nu\Psi, \tag{5.4}$$

[1] $g_{\mu\nu} = \eta_{\mu\nu}$ in case of Minkowski signature.

from where it can be seen that $\theta^{\mu\nu}$ does act as a vielbein in that case. This is also true for the effective metric

$$G^{\mu\nu} = \theta^{\mu\alpha}\theta^{\nu\beta}\eta_{\alpha\beta} = e^{\mu}_{\alpha}\, e^{\nu}_{\beta}\, \eta^{\alpha\beta}. \tag{5.5}$$

The rescaled effective metric is

$$\widetilde{G}^{\mu\nu}_{(\tau)} = e^{-\tau}\theta^{\mu\alpha}\theta^{\nu\beta}\eta_{\alpha\beta}, \tag{5.6}$$

where the scaling factor is found to be

$$e^{-\tau} = |G_{\mu\nu}|^{-1/6}, \tag{5.7}$$

with the relation

$$|\widetilde{G}_{(\tau)\mu\nu}| = |G_{\mu\nu}|^{1/3}. \tag{5.8}$$

Note that here we do not obtain an unimodular metric (at tree level) in contrast to the case of scalars in four dimensions.

The squared Dirac operator takes the following form

$$\begin{aligned} \not{D}^2\Psi &= \gamma_\mu\gamma_\nu[X^\mu,[X^\nu,\Psi]] \\ &\sim -\gamma_\mu\gamma_\nu\theta^{\mu\alpha}\partial_\alpha\left(\theta^{\nu\beta}\partial_\nu\Psi\right) \\ &= -G^{\mu\nu}\partial_\mu\partial_\nu\Psi - a^\mu\partial_\mu\Psi \end{aligned} \tag{5.9}$$

with

$$a^\sigma = \gamma_\mu\gamma_\rho\theta^{\mu\nu}(\partial_\nu\theta^{\rho\sigma}) = -2\mathrm{i}\Sigma_{\alpha\beta}\theta^{\alpha\mu}\partial_\mu\theta^{\beta\rho} + \eta_{\alpha\beta}\theta^{\alpha\mu}\partial_\mu\theta^{\beta\rho}. \tag{5.10}$$

Notice that contrary to the case of Eq. (4.24) the term $\gamma_\mu(\partial_\nu\gamma_\rho)\theta^{\mu\nu}\theta^{\rho\sigma}$ vanishes here because the γ-matrices are not x-dependent. The quadratic form in terms of the properly rescaled metric is

$$S_{\text{square}} = \int \mathrm{d}^x\sqrt{|\widetilde{G}_{\mu\nu}|}\bar{\Psi}\widetilde{\not{D}}^2\Psi = -\int \mathrm{d}^4x\,\bar{\Psi}\left(\widetilde{G}^{\mu\nu}\partial_\mu\partial_\nu\Psi + e^{-\sigma}a^\mu\partial_\nu\Psi\right), \tag{5.11}$$

where the rescaled metric $\widetilde{G}_{\mu\nu}$ is unimodular, i.e. $|\widetilde{G}_{\mu\nu}| = 1$, see Sect. 3.1.2. The one-loop effective action writes then according to Sect. 4.2 as

$$\Gamma_\Psi = \frac{1}{16\,\pi^2}\int \mathrm{d}^4x\left(2\mathrm{tr}(\mathbb{1})\,\Lambda^4 + \mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\mathbb{1} + \mathcal{E}\right)\Lambda^2 + O(\log\Lambda)\right), \tag{5.12}$$

where $\mathrm{tr}(\mathbb{1}) = 4$ for a Dirac fermion. In the above equation the Ricci scalar is expressed in terms of the unimodular metric $\widetilde{G}_{\mu\nu}$, which can be written in terms of $G_{\mu\nu}$ using

$$\begin{aligned} R[\widetilde{G}] &= e^{-\sigma}\left(R[G] - 3\Delta_G\sigma - \frac{3}{2}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)\right), \\ \Delta_G\sigma &= G^{\mu\nu}\partial_\mu\partial_\nu\sigma - \Gamma^\mu\partial_\mu\sigma. \end{aligned} \tag{5.13}$$

We compare the above fermionic effective action Γ_Ψ to the induced action due to a scalar field which we derived in Sect. 3.3.1,

$$\Gamma_\Phi = \frac{1}{16\,\pi^2}\int \mathrm{d}^4x \left(2\Lambda^4 + \frac{R[\widetilde{G}]}{6}\Lambda^2 + O(\log\Lambda)\right). \tag{5.14}$$

Hence we find the following relation

$$\Gamma_\Psi + 4\Gamma_\Phi = \frac{1}{16\,\pi}\int \mathrm{d}^4x\,\mathrm{tr}\mathcal{E}\Lambda^2. \tag{5.15}$$

This equation expresses the cancellation of the induced actions due to fermions and scalars, apart from the $\mathcal{E}$-term. We will come back to this cancellation in Sect. 5.4.

5.1.1 Evaluation of tr$\mathcal{E}$ for $D = 4$

The evaluation of tr$\mathcal{E}$ in the four-dimensional case is much simpler than before. We repeat this computation in different manner in order to have an independent second evaluation at least for the case of a flat background $g_{\mu\nu}$. By making use of the Jacobi identity,

$$\partial_\rho\theta^{-1}_{\mu\nu} + \partial_\nu\theta^{-1}_{\rho\mu} + \partial_\mu\theta^{-1}_{\nu\rho} = 0, \tag{5.16}$$
$$-(\partial_\sigma\theta^{-1}_{\rho\lambda})\left(\theta^{\lambda\mu}\theta^{\sigma\nu} - \theta^{\lambda\nu}\theta^{\sigma\mu}\right) = \partial_\rho\theta^{\mu\nu} \tag{5.17}$$

several terms appearing in the computation of tr$\mathcal{E}$ and $R[\widetilde{G}]$ are equivalent[2]:

$$\begin{aligned}
G^{\mu\nu}(\partial_\nu\theta^{\rho\alpha})(\partial_\rho\theta^{-1}_{\mu\alpha}) &= \frac{1}{2}G^{\mu\nu}(\partial_\mu\theta^{\rho\sigma})(\partial_\nu\theta^{-1}_{\rho\sigma})\\
(\theta^{\rho\sigma}\partial_\mu\theta^{-1}_{\rho\sigma})\theta^{\mu\alpha}G^{\kappa\lambda}(\partial_\kappa\theta^{-1}_{\lambda\alpha}) &= 2(\partial_\mu\theta^{\mu\alpha})G^{\nu\lambda}(\partial_\nu\theta^{-1}_{\lambda\alpha})\\
(\theta^{\rho\sigma}\partial_\mu\theta^{-1}_{\rho\sigma})\theta^{\mu\alpha}(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} &= 2(\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta}\\
\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} &= -2G^{\mu\rho}G^{\nu\sigma}\theta^{-1}_{\mu\alpha}(\partial_\rho\partial_\sigma\theta^{-1}_{\nu\beta})g_{\alpha\beta}\\
\theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} &= G^{\mu\nu}\partial_\mu\partial_\nu\sigma + \theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})(\partial_\nu\sigma)g_{\alpha\beta}\\
&= \frac{1}{2}\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} + \frac{1}{2}G^{\rho\sigma}(\partial_\rho\theta^{\mu\nu})(\partial_\sigma\theta^{-1}_{\mu\nu})\\
&\quad + (\partial_\rho\theta^{\rho\alpha})G^{\sigma\lambda}(\partial_\sigma\theta^{-1}_{\lambda\alpha}).
\end{aligned} \tag{5.18}$$

tr$\mathcal{E}$ was given as

$$\mathrm{tr}\mathcal{E} = -\mathrm{tr}\left(\widetilde{G}^{\mu\nu}\partial_\mu\Omega_\nu + \widetilde{G}^{\mu\nu}\Omega_\mu\Omega_\nu - \widetilde{G}^{\mu\nu}\widetilde{\Gamma}^\rho_{\mu\nu}\Omega_\rho,\right) \tag{5.19}$$

[2]The equations are actually also valid if one considers extra dimensions. Moreover, by means of these relations one can check that the actions Eq. (4.4) and (5.1) are indeed hermitian.

where Ω_μ is now given by

$$\begin{aligned}\Omega_\mu &= \frac{1}{2}\widetilde{G}_{\mu\nu}\left(e^{-\sigma}a^\nu + \widetilde{\Gamma}^\nu\right)\\ &= \frac{1}{2}\left(\gamma_\alpha\gamma_\beta G_{\mu\nu}\theta^{\rho\alpha}(\partial_\rho\theta^{\nu\beta}) - G_{\mu\nu}(\partial_\rho G^{\rho\nu} + \partial_\mu\sigma).\right)\end{aligned} \tag{5.20}$$

Recall also

$$\begin{aligned}\operatorname{tr}\gamma^\alpha\gamma^\beta &= 4g^{\alpha\beta}\\ \operatorname{tr}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta &= 4\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right).\end{aligned} \tag{5.21}$$

For the explicit evaluation of tr$\mathcal{E}$ we use relations stated in Appendix D. We find for the individual parts the following results.

$$\begin{aligned}
\operatorname{tr}\widetilde{G}^{\mu\nu}\partial_\mu\Omega_\nu &= 2e^{-\sigma}\Big\{ + G^{\mu\nu}\left(\partial_\mu G_{\nu\rho}\right)\theta^{\sigma\alpha}\left(\partial_\sigma\theta^{\rho\beta}\right)g_{\alpha\beta}\\
&\quad + \left(\partial_\mu\theta^{\sigma\alpha}\right)\left(\partial_\sigma\theta^{\mu\beta}\right)g_{\alpha\beta} + \theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta}\\
&\quad - G^{\mu\nu}\left(\partial_\mu G_{\nu\rho}\right)\left(\partial_\sigma G^{\rho\sigma}\right) - \partial_\mu\partial_\nu G^{\mu\nu} + G^{\mu\nu}\partial_\mu\partial_\nu\sigma\Big\}\\
&= 2e^{-\sigma}\Big\{\left(\partial_\mu\theta^{\mu\alpha}\right)G^{\nu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right) - \theta^{\mu\alpha}\partial_\mu\partial_\nu\theta^{\nu\beta}g_{\alpha\beta}\\
&\quad + \frac{1}{2}G^{\rho\sigma}\left(\partial_\rho\theta^{\mu\nu}\right)\left(\partial_\sigma\theta^{-1}_{\mu\nu}\right) + \frac{1}{2}\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu}\Big\}\\
&= 0\\
\operatorname{tr}\widetilde{G}^{\mu\nu}\Omega_\mu\Omega_\nu &= e^{-\sigma}\Big\{\left(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}\right)G_{\nu\kappa}\theta^{\sigma\alpha}\left(\partial_\sigma\theta^{\nu\beta}\right)\theta^{\lambda\gamma}\left(\partial_\lambda\theta^{\kappa\delta}\right)\\
&\quad - 2\left(\partial_\sigma G^{\sigma\nu}\right)G_{\nu\kappa}g_{\gamma\delta}\theta^{\lambda\gamma}\left(\partial_\lambda\theta^{\kappa\delta}\right) + 2\theta^{\sigma\alpha}\left(\partial_\sigma\theta^{\nu\beta}\right)\left(\partial_\nu\sigma\right)g_{\alpha\beta}\\
&\quad - 2\left(\partial_\lambda G^{\lambda\kappa}\right)\left(\partial_\kappa\sigma\right) + \left(\partial_\sigma G^{\sigma\nu}\right)\left(\partial_\lambda G^{\lambda\kappa}\right)G_{\nu\kappa} + G^{\mu\nu}\left(\partial_\mu\sigma\right)\left(\partial_\nu\sigma\right)\Big\}\\
&= e^{-\sigma}\Big\{ - G^{\kappa\lambda}G^{\mu\nu}\left(\partial_\kappa\theta^{-1}_{\mu\alpha}\right)\left(\partial_\lambda\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta}\\
&\quad + G^{\mu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right)G^{\nu\lambda}\left(\partial_\lambda\theta^{-1}_{\mu\beta}\right)g^{\alpha\beta}\Big\}\\
\operatorname{tr}\Omega_\mu\widetilde{\Gamma}^\mu &= e^{-\sigma}\mathrm{Tr}\left(\Omega_\mu\Gamma^\mu - \Omega_\mu G^{\mu\nu}\partial_\nu\sigma\right)\\
&= e^{-\sigma}\mathrm{Tr}\left(-\Omega_\mu\left(\partial_\nu G^{\mu\nu}\right) + \Omega_\mu G^{\mu\nu}\partial_\nu\sigma\right)\\
&= 0
\end{aligned} \tag{5.22}$$

Our final result for tr$\mathcal{E}$ in four spacetime dimensions is thus

$$\operatorname{tr}\mathcal{E} = e^{-\sigma}\Big\{G^{\kappa\lambda}G^{\mu\nu}\left(\partial_\kappa\theta^{-1}_{\mu\alpha}\right)\left(\partial_\lambda\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta} - G^{\mu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right)G^{\nu\lambda}\left(\partial_\lambda\theta^{-1}_{\mu\beta}\right)g^{\alpha\beta}\Big\}. \tag{5.23}$$

5.1.2 Evaluation of $R[\widetilde{G}]$ for $D = 4$

Remember that the Ricci in terms of the metric was found to be

$$\begin{aligned} R[G] = {} & (\partial_\beta G^{\beta\delta})G^{\gamma\alpha}(\partial_\alpha G_{\gamma\delta}) + G^{\alpha\beta}G^{\gamma\delta}\partial_\beta\partial_\delta G_{\alpha\gamma} - G^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma G_{\mu\nu} \\ & - (\partial_\beta G^{\beta\delta})G^{\mu\nu}(\partial_\delta G_{\mu\nu}) - \frac{3}{4}G^{\rho\sigma}(\partial_\rho G^{\mu\nu})(\partial_\sigma G^{\mu\nu}) \\ & + \frac{1}{2}G^{\nu\rho}(\partial_\rho G^{\alpha\gamma})(\partial_\gamma G_{\alpha\nu}) - \frac{1}{4}G^{\rho\sigma}G^{\mu\nu}(\partial_\rho G_{\mu\nu})G^{\kappa\lambda}(\partial_\sigma G_{\kappa\lambda}). \end{aligned} \tag{5.24}$$

Using relations given in Appendix D we rewrite the Ricci now in terms of the Poisson structure $\theta^{\mu\nu}$ which in four dimensions has the function of a vielbein. Using the explicit formula for the metric $G^{\mu\nu}(x)$,

$$G^{\mu\nu} = \theta^{\mu\alpha}\theta^{\nu\beta}g_{\alpha\beta}, \tag{5.25}$$

the Ricci becomes

$$\begin{aligned} R = {} & -2\left(\partial_\alpha\theta^{\alpha\rho}\right)G^{\beta\gamma}\left(\partial_\gamma\theta^{-1}_{\beta\rho}\right) - \left(\partial_\alpha\theta^{\alpha\rho}\right)\left(\partial_\beta\theta^{\beta\sigma}\right)g_{\rho\sigma} \\ & + 2G^{\mu\rho}G^{\nu\sigma}\theta^{-1}_{\mu\alpha}\partial_\rho\partial_\sigma\theta^{-1}_{\nu\beta}g^{\alpha\beta} + \frac{1}{2}G^{\beta\gamma}\left(\partial_\gamma\theta^{-1}_{\alpha\rho}\right)G^{\alpha\delta}\left(\partial_\delta\theta^{-1}_{\beta\sigma}\right)g^{\rho\sigma} \\ & + 2\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} - \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}\left(\partial_\rho\theta^{-1}_{\mu\alpha}\right)\left(\partial_\sigma\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta} \\ & + 4\theta^{\beta\gamma}\left(\partial_\beta\theta^{\delta\alpha}\right)\left(\partial_\delta\sigma\right)g_{\gamma\alpha} + \frac{3}{2}G^{\rho\sigma}\left(\partial_\rho\theta^{\mu\nu}\right)\left(\partial_\sigma\theta^{-1}_{\mu\nu}\right) \\ & - G^{\mu\rho}\left(\partial_\rho\theta^{\nu\sigma}\right)\left(\partial_\nu\theta^{-1}_{\mu\sigma}\right) \quad - \frac{1}{2}\left(\partial_\nu\theta^{\gamma\sigma}\right)\left(\partial_\gamma\theta^{\nu\rho}\right)g_{\rho\sigma}. \end{aligned} \tag{5.26}$$

This equation holds in fact for any vielbein using the identification

$$\theta^{\mu\nu}g_{\nu\alpha} = e^{\ \mu}_{\alpha} \qquad \theta^{-1}_{\mu\nu}g^{\nu\alpha} = -e^{\ \alpha}_{\mu} \tag{5.27}$$

since in the above derivation we have not exploited any property of the Poisson structure $\theta^{\mu\nu}$. By making use of the Jacobi relation of Eq. (5.16) we can reduced $R[G]$ to the following expression

$$\begin{aligned} R[G] = {} & \theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} + G^{\rho\sigma}\left(\partial_\rho\theta^{\mu\nu}\right)\left(\partial_\sigma\theta^{-1}_{\mu\nu}\right) \\ & + 2\left(\partial_\mu\theta^{\mu\alpha}\right)G^{\nu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right) - G^{\mu\nu}\left(\partial_\mu\sigma\right)\left(\partial_\nu\sigma\right) \\ & + \frac{1}{2}G^{\mu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right)G^{\nu\lambda}\left(\partial_\lambda\theta^{-1}_{\mu\beta}\right)g^{\alpha\beta} - \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}\left(\partial_\rho\theta^{-1}_{\mu\alpha}\right)\left(\partial_\sigma\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta} \\ & - \frac{1}{2}\left(\partial_\mu\theta^{\nu\alpha}\right)\left(\partial_\nu\theta^{\mu\beta}\right)g_{\alpha\beta}. \end{aligned} \tag{5.28}$$

Of course, we are interested in the Ricci scalar with respect to the rescaled metric $\widetilde{G}^{\mu\nu}$. The relationship between $R[G]$ and $R[\widetilde{G}]$ is given by [57]

$$R[\widetilde{G}] = e^{-\sigma}\left(R[G] - 3\Delta_G\sigma - \frac{3}{2}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)\right). \tag{5.29}$$

Evaluating also

$$
\begin{aligned}
-3\Delta_G\sigma - \frac{3}{2}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) = &-\frac{3}{2}G^{\mu\nu}(\partial_\mu\theta^{\rho\sigma})(\partial_\nu\theta^{-1}_{\rho\sigma}) - \frac{3}{2}\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} \\
&+ \frac{3}{2}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) - 3(\partial_\mu\theta^{\mu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\alpha})
\end{aligned} \tag{5.30}
$$

gives

$$
\begin{aligned}
R[\widetilde{G}] = e^{-\sigma}\Big[&-\frac{1}{2}\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} - \frac{1}{2}G^{\rho\sigma}\left(\partial_\rho\theta^{\mu\nu}\right)\left(\partial_\sigma\theta^{-1}_{\mu\nu}\right) \\
&- \left(\partial_\mu\theta^{\mu\alpha}\right)G^{\nu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right) + \frac{1}{2}G^{\mu\nu}\left(\partial_\mu\sigma\right)\left(\partial_\nu\sigma\right) \\
&+ \frac{1}{2}G^{\mu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right)G^{\nu\lambda}\left(\partial_l\theta^{-1}_{\mu\beta}\right)g^{\alpha\beta} \\
&- \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}\left(\partial_\rho\theta^{-1}_{\mu\alpha}\right)\left(\partial_\sigma\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta} - \frac{1}{2}\left(\partial_\mu\theta^{\nu\alpha}\right)\left(\partial_\nu\theta^{\mu\beta}\right)g_{\alpha\beta}\Big].
\end{aligned} \tag{5.31}
$$

Via partial integration, the number of independent terms can be further reduced:

$$
\begin{aligned}
\int \mathrm{d}^4x\, e^{-\sigma}\theta^{\mu\alpha}\partial_\mu\partial_\nu\theta^{\nu\beta}g_{\alpha\beta} &= 0, \\
\int \mathrm{d}^4x\, e^{-\sigma}\left(\partial_\mu\theta^{\nu\alpha}\right)\left(\partial_\nu\theta^{\mu\beta}\right)g_{\alpha\beta} &= -\int \mathrm{d}^4x\, \frac{e^{-\sigma}}{2}\Big(\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} \\
&\qquad + G^{\rho\sigma}\left(\partial_\rho\theta^{\mu\nu}\right)\left(\partial_\sigma\theta^{-1}_{\mu\nu}\right)\Big), \\
\int \mathrm{d}^4x\, e^{-\sigma}\left(\partial_\rho\theta^{\rho\alpha}\right)G^{\sigma\kappa}\left(\partial_\kappa\theta^{-1}_{\sigma\kappa}\right) &= \int \mathrm{d}^4x\, e^{-\sigma}\left(\partial_\mu\theta^{\nu\alpha}\right)\left(\partial_\nu\theta^{\mu\beta}\right)g_{\alpha\beta}.
\end{aligned} \tag{5.32}
$$

We yield the following compact form for the Einstein-Hilbert action for the unimodular metric $\widetilde{G}^{\mu\nu}$:

$$
\begin{aligned}
\int \mathrm{d}^4x\, R[\widetilde{G}] = \int \mathrm{d}^4x\, e^{-\sigma}\Bigg(&\frac{1}{2}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
&- \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&- \frac{1}{2}(\partial_\rho\theta^{\rho\alpha})G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha}) + \frac{1}{2}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)\Bigg).
\end{aligned} \tag{5.33}
$$

Comparing with (5.23), we can write this as

$$
\int \mathrm{d}^4x\, \mathrm{tr}\,\mathcal{E} = \int \mathrm{d}^4x\, \left(-2\,R[\tilde{G}] - \left(\partial_\rho\theta^{\rho\alpha}\right)G^{\sigma\kappa}\left(\partial_\kappa\theta^{-1}_{\sigma\alpha}\right) + G^{\mu\nu}\left(\partial_\mu\sigma\right)\left(\partial_\nu\sigma\right)\right). \tag{5.34}
$$

This formula applies for Dirac fermions, and with an additional factor 1/2 for Weyl fermions. The middle term in the above equation vanishes for on-shell geometries which fulfill

$$
G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\alpha}) = 0. \tag{5.35}
$$

Our final results are hence

$$\int \mathrm{d}^4x \; \mathrm{tr}\, \mathcal{E} \overset{\text{eom}}{=} \int \mathrm{d}^4x \; \left(-2R[\tilde{G}] + G^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right) \tag{5.36}$$

and

$$\Gamma_\Psi = \int \mathrm{d}^4x \,\mathrm{tr}\mathbb{1} \left(2\Lambda^4 - \frac{1}{6}R[\widetilde{G}]\Lambda^2 + \frac{1}{4}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)\,\Lambda^2 + O(\log(\Lambda)\right). \tag{5.37}$$

Comparison with previous results. The result in four dimensions is somewhat nicer than the general result Eq. (4.70) we had before since we do not get the term $R[g]_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$ which breaks Lorentz invariance. In the above result Eq. (5.36) we have exploited relations from partial integration and the e.o.m.. However, in the previous computations of Sect. 4.4 we made use of the e.o.m. only. It turns out that some of the relations of partial integration of Eq. (5.32) are equivalent to the e.o.m. because of the following relations:

$$\begin{aligned}
\theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})g_{\alpha\beta} &= 0\\
(\partial_\nu\theta^{\mu\alpha})(\partial_\mu\theta^{\nu\beta})g_{\alpha\beta} &= -\theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta}\\
&= -\frac{1}{2}G^{\mu\nu}\theta^{\rho\sigma}\partial_\mu\partial_\nu\theta^{-1}_{\rho\sigma} - \frac{1}{2}G^{\mu\nu}(\partial_\mu\theta^{\rho\sigma})(\partial_\nu\theta_{\rho\sigma}),
\end{aligned} \tag{5.38}$$

where we have used the Jacobi relations Eq. (5.18) in the last step. The Ricci scalar of Eq. (5.31) for on-shell geometries takes thus the form

$$\begin{aligned}
R[\widetilde{G}] \overset{\text{e.o.m.}}{=} e^{-\sigma}\Big[\, &\frac{1}{2}G^{\mu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right)G^{\nu\lambda}\left(\partial_l\theta^{-1}_{\mu\beta}\right)g^{\alpha\beta} - \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}\left(\partial_\rho\theta^{-1}_{\mu\alpha}\right)\left(\partial_\sigma\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta}\\
&+\frac{1}{2}\left(\partial_\mu\theta^{\nu\alpha}\right)\left(\partial_\nu\theta^{\mu\beta}\right)g_{\alpha\beta} + \frac{1}{2}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)\Big].
\end{aligned} \tag{5.39}$$

This is in agreement with the Ricci scalar of Eq. (4.57) for a constant background metric. Also, $\mathrm{tr}\mathcal{E}$ is in accordance with our previous result quoted in Eq. (4.52) if one considers on-shell geometries and a constant $g_{\mu\nu}$.

5.2 Interpretation as gauge theory on $\mathbb{R}^4_\theta$

Now we want to interpret the geometrical action Eq. (5.1) for fermions as an action for a Dirac fermion on Moyal-Weyl space $\mathbb{R}^4_\theta$ coupled to a $U(1)$ gauge field in the adjoint. This point of view is obtained by writing the general covariant coordinate matrix X^α as

$$X^\alpha = \bar{X}^\alpha + \mathcal{A}^\alpha\,. \tag{5.40}$$

Here $\bar{X}^\alpha$ are generators of the Moyal-Weyl quantum plane, which satisfy

$$[\bar{X}^\alpha, \bar{X}^\beta] = \mathrm{i}\bar{\theta}^{\alpha\beta}\,, \tag{5.41}$$

where $\bar{\theta}^{\alpha\beta}$ is a constant antisymmetric tensor. These are particular solutions of the equations of motion Eq. (3.41). The matrices $\bar{X}^\alpha$ are the quantization of the coordinate functions $\bar{x}^\alpha$ of Moyal-Weyl space. The effective geometry for the Moyal-Weyl plane is indeed flat, given by

$$\bar{g}^{\alpha\beta} = \bar{\theta}^{\alpha\gamma}\,\bar{\theta}^{\beta\delta} g_{\gamma\delta}, \tag{5.42}$$

which can be brought into unimodular form again

$$\begin{aligned} \tilde{g}^{\alpha\beta} &= \bar{\rho}\,\bar{g}^{\alpha\beta}, \\ \bar{\rho} &= (\det\bar{\theta}^{\alpha\beta})^{-1/2} = |\bar{g}_{\alpha\beta}|^{1/4} \equiv \Lambda_{\mathrm{NC}}^4, \\ \det(\tilde{g}^{\alpha\beta}) &= 1. \end{aligned} \tag{5.43}$$

Consider now the change of variables

$$\mathcal{A}^\alpha(\bar{x}) = -\bar{\theta}^{\alpha\beta} A_\beta(\bar{x}) \tag{5.44}$$

where $A_\beta(\bar{x})$ are hermitian matrices, interpreted as smooth functions on $\mathbb{R}^4_{\bar{\theta}}$. Thus we can write

$$[X^\alpha, f] = \left[\bar{X}^\alpha + \mathcal{A}^\alpha, f\right] = \mathrm{i}\bar{\theta}^{\alpha\beta}\left(\frac{\partial}{\partial\bar{x}^\beta} f + i\,[A_\beta, f]\right) \equiv i\bar{\theta}^{\alpha\beta} D_\beta f, \tag{5.45}$$

giving for the quadratic form Eq. (5.11)

$$\begin{aligned} S_{\mathrm{square}} &= (2\pi)^2\,\mathrm{Tr}\,\Psi^\dagger\gamma_\alpha\gamma_\beta\left[X^\alpha, \left[X^\beta, \Psi\right]\right] \\ &= -\int \mathrm{d}^4\bar{x}\,\bar{\rho}\,\Psi^\dagger\,\gamma_\alpha\gamma_\beta\bar{\theta}^{\alpha\mu}\bar{\theta}^{\beta\nu} D_\mu D_\nu \Psi \\ &= \int \mathrm{d}^4\bar{x}\,\Psi^\dagger\,\widetilde{\not{D}^2}_A\Psi\,. \end{aligned} \tag{5.46}$$

This is an exact expression on $\mathbb{R}^4_{\bar{\theta}}$, where

$$\widetilde{\not{D}^2}_A = -\bar{\rho}\,\gamma_\alpha\gamma_\beta\bar{\theta}^{\alpha\mu}\bar{\theta}^{\beta\nu} D_\mu D_\nu \;= -\,\bar{\gamma}^\mu\bar{\gamma}^\nu\, D_\mu D_\nu\,. \tag{5.47}$$

$\bar{\gamma}^\alpha$ are the elements of the Clifford algebra associated to $\tilde{g}^{\alpha\beta}$,

$$\begin{aligned} \bar{\gamma}^\alpha &= (\det\bar{g}_{\alpha\beta})^{\frac{1}{8}}\,\gamma_\beta\,\bar{\theta}^{\beta\alpha}, \\ \{\bar{\gamma}^\alpha, \bar{\gamma}^\beta\} &= 2\,\tilde{g}^{\alpha\beta}\,. \end{aligned} \tag{5.48}$$

Coordinate transformation $x^\mu \to \bar{x}^\mu$. We want to rewrite the geometrical results of Sect. 5.1 in terms of a gauge theory on $\mathbb{R}^4_{\bar{\theta}}$ in $\bar{x}$-coordinates. Let us first give the correct transformation rules between the coordinates $X^\mu \sim x^\mu$, and the coordinates $\bar{X}^\mu \sim \bar{x}^\mu$. The leading-order relation between the coordinates x and $\bar{x}$ follows from Eq. (5.40)

$$x^\mu = \bar{x}^\mu - \bar{\theta}^{\mu\nu}\bar{A}_\nu + O(\bar{\theta}^2), \tag{5.49}$$

with the Jacobian

$$\begin{aligned}\left|\frac{\partial x^\mu}{\partial \bar{x}^\nu}\right| &= 1 - \bar{\theta}^{\mu\alpha}\frac{\partial \bar{A}_\alpha}{\partial \bar{x}^\nu} + O(\bar{\theta}^2)\\ &= 1 - \frac{1}{2}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu} + O(\bar{\theta}^2).\end{aligned} \tag{5.50}$$

For a notation as clear as possible we will denote all $\bar{x}$-dependent tensors with a bar, e.g.

$$\bar{F}_{\mu\nu} = \bar{\partial}_\mu \bar{A}_\nu - \bar{\partial}_\nu \bar{A}_\mu + \mathrm{i}[\bar{A}_\mu, \bar{A}_\nu], \tag{5.51}$$

and we distinguish

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \bar{\partial}_\mu = \frac{\partial}{\partial \bar{x}^\mu}. \tag{5.52}$$

The non-constant Poisson tensor writes in terms of the $U(1)$ field strength as

$$\mathrm{i}\theta^{\mu\nu}(x) = \mathrm{i}\bar{\theta}^{\mu\nu} - \mathrm{i}\bar{\theta}^{\mu\alpha}\bar{\theta}^{\nu\beta}\bar{F}_{\alpha\beta}, \tag{5.53}$$

and the inverse Poisson tensor amounts to

$$\theta^{-1}_{\mu} = \bar{\theta}^{-1}_{\mu\nu} - \bar{F}_{\mu\nu} + O(\bar{\theta}^2). \tag{5.54}$$

We will also need the metric $G^{\mu\nu}(x)$

$$\begin{aligned}G^{\alpha\beta} &= (\bar{\theta}^{\alpha\gamma} - \bar{\theta}^{\alpha\mu}\bar{\theta}^{\gamma\nu}\bar{F}_{\mu\nu})(\bar{\theta}^{\beta\delta} - \bar{\theta}^{\beta\rho}\bar{\theta}^{\delta\sigma}\bar{F}_{\rho\sigma})g_{\gamma\delta}\\ &= \bar{g}^{\alpha\gamma}\left(\delta^\beta_\gamma + \bar{F}_{\gamma\delta}\bar{\theta}^{\delta\beta} + \bar{g}_{\gamma\mu}\bar{\theta}^{\mu\nu}\bar{F}_{\nu\delta}\bar{g}^{\delta\beta} + \bar{g}_{\gamma\delta}\bar{\theta}^{\delta\mu}\bar{F}_{\mu\nu}\bar{g}^{\nu\rho}\bar{F}_{\rho\sigma}\bar{\theta}^{\sigma\beta}\right)\\ &\equiv \bar{g}^{\alpha\gamma}\left(\delta^\beta_\gamma + M^\beta_\gamma\right),\end{aligned} \tag{5.55}$$

as well as $e^\sigma = |G^{\mu\nu}|^{1/4}$ in terms of $\bar{x}^\mu$. To compute the determinant, we use

$$\det(\mathbb{1} + M) = 1 + \mathrm{tr}M + \frac{1}{2}((\mathrm{tr}M)^2 - \mathrm{tr}(M^2)) + O(M^3). \tag{5.56}$$

We see that

$$\mathrm{tr}M = -2\bar{F}_{\mu\nu}\bar{\theta}^{\mu\nu} - \bar{\theta}^{\alpha\mu}\bar{g}_{\mu\nu}\bar{\theta}^{\beta\nu}\bar{F}_{\alpha\rho}\bar{g}^{\rho\sigma}\bar{F}_{\sigma\beta}. \tag{5.57}$$

For the density factor this means

$$\rho(x)^{-1} = e^{\sigma} = |\bar{g}^{\alpha\beta}|^{1/4} \left(1 - \frac{1}{2}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu} + O(\bar{\theta}^3)\right). \tag{5.58}$$

By a straightforward application of the above relations one can write the second Seeley-de Witt coefficient in $\bar{x}$-coordinates. In Appendix E this expansion is given explicitly for the individual terms. We have omitted $O(\bar{A})$ terms from both $R[\widetilde{G}]$ and tr$\mathcal{E}$, which are total derivatives and do not contribute to the effective action. Let us only quote the results here

$$\int \mathrm{d}^4x \; R[\widetilde{G}] = \int \mathrm{d}^4x \; \rho(x) \left(\frac{1}{2}\bar{\partial}^2\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu} + \frac{1}{4}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\bar{F}_{\nu\mu}\bar{\partial}^2\bar{F}_{\beta\alpha}\right), \tag{5.59}$$

which agrees with Eq. (78) in [48], as it should be. tr$\mathcal{E}$ in $\bar{x}$-coordinates is given as

$$\int \mathrm{d}^4x \; \mathrm{tr}\mathcal{E} = -\frac{1}{2}\int \mathrm{d}^4x|\bar{g}_{\alpha\beta}|^{1/4} \; \bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}\bar{F}_{\alpha\beta}\bar{\partial}^2\bar{F}_{\gamma\delta}, \tag{5.60}$$

where

$$\bar{\partial}^2 = \bar{\partial}_\alpha\bar{\partial}^\alpha. \tag{5.61}$$

We find for the one-loop induced action the following result

$$\begin{aligned}
\Gamma_\Psi &= \frac{1}{16\pi^2}\int \mathrm{d}^4x \left(2\,\mathrm{tr}(\mathbb{1})\Lambda^4 + \mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\mathbb{1} + \mathcal{E}\right)\Lambda^2 + O(\log\Lambda)\right) \\
&= -4\Gamma_\Phi + \frac{1}{16\pi^2}\int \mathrm{d}^4x \; \mathrm{tr}\mathcal{E} \; \Lambda^2 \\
&= -4\Gamma_\Phi - \frac{1}{16\pi^2}\int \mathrm{d}^4x \; \frac{\rho(x)}{2} \; \bar{g}^{\alpha\gamma} \; \bar{g}^{\beta\delta}\bar{F}_{\alpha\beta}\bar{\partial}^2\bar{F}_{\gamma\delta}\Lambda^2,
\end{aligned} \tag{5.62}$$

where we used Eq. (5.15) and

$$|\bar{g}_{\alpha\beta}|^{1/4} \; \bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}\bar{F}_{\alpha\beta}\bar{\partial}^2\bar{F}_{\gamma\delta} = \rho(x) \; \bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}\bar{F}_{\alpha\beta}\bar{\partial}^2\bar{F}_{\gamma\delta} \tag{5.63}$$

to order $O(\bar{A}^2)$. It remains to be discussed that there is a nontrivial relation between the cutoff Λ of the geometrical action and the cutoff $\bar{\Lambda}$ of the $U(1)$ gauge theory, which follows from the identity

$$S_{\mathrm{square}} = \mathrm{Tr}\;\Psi^\dagger\gamma_\alpha\gamma_\beta\left[X^\alpha, \left[X^\beta, \Psi\right]\right] = \int \mathrm{d}^4x \; \Psi^\dagger \; \widetilde{\not{D}}^2_{\widetilde{G}} \; \Psi = \int \mathrm{d}^4x \; \frac{\rho(x)}{\bar{\rho}} \; \Psi^\dagger \widetilde{\not{D}}^2_{\bar{A}}\Psi. \tag{5.64}$$

For the Laplacians this means

$$\widetilde{\not{D}}^2_{\widetilde{G}} = \frac{\rho(x)}{\bar{\rho}} \; \widetilde{\not{D}}^2_{\bar{A}}. \tag{5.65}$$

Since we have implemented the cutoffs using the Schwinger parameterizations in Eq. (4.29), the cutoffs are related as follows

$$\Lambda^2 = \frac{\rho(x)}{\bar{\rho}}\,\bar{\Lambda}^2. \tag{5.66}$$

This makes sense provided $\rho(x)/\bar{\rho}$ varies only on large scales respectively small momenta $p \ll \bar{\Lambda}$ which we take as our working assumption. Together with Eq. (5.50), we obtain as a final result for the geometric one-loop effective action expressed in terms of gauge thoery on $\mathbb{R}^4_\theta$

$$\begin{aligned}\Gamma_\Psi &= -4\Gamma_\Phi - \int \mathrm{d}^4\bar{x}\,\frac{\bar{\rho}}{2}\,\bar{g}^{\alpha\gamma}\,\bar{g}^{\beta\delta}\,\bar{F}_{\alpha\beta}\,\bar{\partial}^2\bar{F}_{\gamma\delta}\,\bar{\Lambda}^2\\ &= -4\Gamma_\Phi + \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\widetilde{g}^{\alpha\gamma}\,\widetilde{g}^{\beta\delta}\,\bar{F}_{\alpha\beta}(p)\,\bar{F}_{\gamma\delta}(-p)\,\frac{p^2}{\Lambda^4_{NC}}\frac{\bar{\Lambda}^2}{2},\end{aligned} \tag{5.67}$$

where $p^2 = p_\mu p_\nu g^{\mu\nu}$. This agrees precisely with the one-loop computation in the gauge theory point of view of Eq. (5.83) obtained below.

5.3 Identifying UV/IR mixing

In this section, we compare the geometrical form of the one-loop effective action obtained in the previous section with the one-loop effective action obtained from the gauge theory point of view. The result is of course the same, which assures that the obtained geometrical one-loop effective action Eq. (5.37) is correct. It also sheds new light on the conditions to which extent the semi-classical analysis of the previous section is valid. This generalizes the results of [48] to the fermionic case. We find as expected that the UV/IR mixing terms obtained by integrating out the fermions are given by the induced geometrical action Eq. (5.37), in a suitable IR regime. In particular, we need an explicit, physical momentum cutoff $\bar{\Lambda}$.

Using the variables and conventions of the previous section, the action Eq. (5.11) can be exactly rewritten as $U(1)$ gauge theory on $\mathbb{R}^4_\theta$, which in the Euclidean case takes the form

$$\begin{aligned}S[\Psi] &= (2\pi)^2\,\mathrm{Tr}\Psi^\dagger\gamma_\alpha[X^\alpha,\Psi]\\ &= \int \mathrm{d}^4\bar{x}\,\tilde{\Psi}^\dagger \mathrm{i}\bar{\gamma}^\alpha(\bar{\partial}_\alpha\tilde{\Psi} + \mathrm{i}g[\bar{A}_\alpha,\tilde{\Psi}])\end{aligned} \tag{5.68}$$

We introduce an explicit coupling constant g, and define a rescaled fermionic field

$$\tilde{\Psi} = |\bar{g}_{\alpha\beta}|^{\frac{1}{16}}\,\Psi \tag{5.69}$$

in order to obtain the properly normalized effective metric $\tilde{g}^{\alpha\beta}$; we will omit the tilde on Ψ henceforth. Recall also that only $U(1)$ gauge fields are considered here, because only those correspond to the nontrivial geometry considered in the previous section.

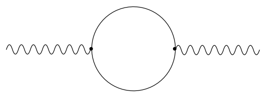

Figure 5.1: Fermionic one-loop diagram.

We need the $O(\bar{A}^2)$ contribution to the one-loop effective action obtained by integrating out the fermionic field Ψ. While this computation has been discussed several times in the literature [20, 81, 82, 83, 84], the known results are not accurate enough for our purpose, i.e. in the regime $p^2 < \Lambda_{\rm NC}^2$ and $\bar{\Lambda}^2 < \Lambda_{\rm NC}^2$. There the semiclassical geometry is expected to make sense. We need to analyze carefully the IR regime of the well-known effective cutoff $\Lambda_{\rm eff}(p)$ (5.75) for non-planar graphs as $p \to 0$, while keeping $\bar{\Lambda}$ fixed. In this regime the non-planar diagrams almost coincide with the planar diagrams, and the leading IR corrections due to the nonplanar diagrams correspond to the induced geometrical terms in Eq. (5.37). This has not been considered in previous attempts to explain UV/IR mixing, e.g. in terms of exchange of closed string modes [85, 86].

To proceed one can either square the Dirac operator as in [84], or use directly the fermionic Feynman rules. We choose the latter approach here, and consider the Feynman diagram in Figure 5.1 corresponding to

$$
\begin{aligned}
\Gamma_\Psi &= -\frac{1}{2}\mathrm{Tr}\log\Delta_0 - \frac{g^2}{2}\left\langle \int \mathrm{d}^4x\,\bar{\rho}\,\bar{\Psi}\tilde{\gamma}^\alpha[\bar{A}_\alpha,\Psi]\int \mathrm{d}^4\bar{x}\,\bar{\rho}\,\bar{\Psi}\bar{\gamma}^\beta[\bar{A}_\beta,\Psi]\right\rangle \\
&= -\frac{1}{2}\mathrm{Tr}\log\Delta_0 + \Gamma_\Psi(\bar{A}),
\end{aligned} \tag{5.70}
$$

where the first term is coming from the interaction free part of the action Eq. (5.68). The minus sign in front is due to the fermionic loop. This gives

$$
\begin{aligned}
\Gamma_\Psi = -4g^2 \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, &\bar{A}_{\alpha'}(p)\bar{A}_{\beta'}(-p)\,\tilde{g}^{\alpha'\alpha}\tilde{g}^{\beta'\beta}\times \\
&\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{2k_\alpha k_\beta + k_\alpha p_\beta + p_\alpha k_\beta - \tilde{g}_{\alpha\beta}k(k+p)}{(k\cdot k)((k+p)\cdot(k+p))}\times \\
&\left(e^{-\mathrm{i}k_\mu\theta^{\mu\nu}p_\nu} - 1\right)
\end{aligned} \tag{5.71}
$$

which is quite close to the bosonic case, using the notation

$$
\begin{aligned}
k\cdot k &\equiv k_\mu k_\nu\,\tilde{g}^{\mu\nu}, \\
k^2 &\equiv k_\mu\,k_\nu g^{\mu\nu}.
\end{aligned} \tag{5.72}
$$

To evaluate this loop integral, we rewrite it in a different way as in [84]

$$
\begin{aligned}
&-\int \frac{d^4k}{(2\pi)^4} \frac{4k_\alpha k_\beta + 2k_\alpha p_\beta + 2p_\alpha k_\beta - 2\tilde{g}_{\alpha\beta}\, k\,(k+p)}{(k\cdot k)((k+p)\cdot(k+p))}\big(e^{-ik_\mu\theta^{\mu\nu}p_\nu} - 1\big) \\
&= -\int \frac{d^4k}{(2\pi)^4} \Big(\frac{(2k_\alpha + p_\alpha)(2k_\beta + p_\beta) - (p_\alpha p_\beta - \tilde{g}_{\alpha\beta} p\cdot p)}{(k\cdot k)((k+p)\cdot(k+p))} \\
&\quad - \tilde{g}_{\alpha\beta}\Big(\frac{1}{k\cdot k} + \frac{1}{(k+p)\cdot(k+p)}\Big)\Big)\big(e^{-ik_\mu\theta^{\mu\nu}p_\nu} - 1\big) \\
&= -\int \frac{d^4k}{(2\pi)^4} \Big(\frac{(2k_\alpha + p_\alpha)(2k_\beta + p_\beta)}{(k\cdot k)((k+p)\cdot(k+p))} - 2\frac{\tilde{g}_{\alpha\beta}}{k\cdot k}\Big)\big(e^{-ik_\mu\theta^{\mu\nu}p_\nu} - 1\big) \\
&\quad + (p_\alpha p_\beta - \tilde{g}_{\alpha\beta} p\cdot p)\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k\cdot k)((k+p)\cdot(k+p))}\big(e^{-ik_\mu\theta^{\mu\nu}p_\nu} - 1\big)
\end{aligned}
\tag{5.73}
$$

where we replaced $\frac{1}{(k+p)\cdot(k+p)}$ by $\frac{1}{k\cdot k}$ under the integral (which does not make a difference in the regularization used here). Now the first term is precisely the induced action Γ_Φ obtained by integrating out a scalar field Φ [48], which is known to be gauge invariant. The second term is logarithmic and manifestly gauge-invariant. Therefore

$$
\begin{aligned}
\Gamma_\Psi &= -4\,\Gamma_\Phi + g^2 n_f \int \frac{d^4p}{(2\pi)^4} A_{\alpha'}(p) A_{\beta'}(-p)\; \tilde{g}^{\alpha'\alpha}\tilde{g}^{\beta'\beta}\,(p_\alpha p_\beta - \tilde{g}_{\alpha\beta} p\cdot p) \\
&\quad \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k\cdot k)((k+p)\cdot(k+p))}\big(e^{-ik_\mu\theta^{\mu\nu}p_\nu} - 1\big) \\
&= -4\,\Gamma_\Phi - g^2 n_f \int \frac{d^4p}{(2\pi)^4} A_{\alpha'}(p) A_{\beta'}(-p)\; \tilde{g}^{\alpha'\alpha}\tilde{g}^{\beta'\beta}\,(p_\alpha p_\beta - \tilde{g}_{\alpha\beta} p\cdot p) \\
&\quad \frac{1}{8\pi^2}\int_0^1 dz\, \Big(K_0\big(2\sqrt{\frac{z(1-z)p\cdot p}{\Lambda^2}}\big) - K_0\big(2\sqrt{\frac{z(1-z)p\cdot p}{\Lambda_{\text{eff}}^2}}\big)\Big)\,,
\end{aligned}
\tag{5.74}
$$

for Dirac fermions, where

$$
\Lambda_{\text{eff}}^2 = \frac{1}{1/\bar{\Lambda}^2 + \frac{1}{4}\frac{p^2}{\Lambda_{\text{NC}}^4}} = \Lambda_{\text{eff}}^2(p)
\tag{5.75}
$$

is the "effective" cutoff for non-planar graphs, and Λ_{NC} is defined in Eq. (5.42). For the standard evaluation of the k-integration see e.g. [48]. To proceed we consider the IR regime

$$
\frac{p^2\bar{\Lambda}^2}{\Lambda_{\text{NC}}^4} < 1\,,
\tag{5.76}
$$

see Figure 5.2. Then both $\bar{\Lambda}$ and Λ_{eff} are large, and we can use the asymptotic expansions

$$
K_0\Big(2\sqrt{\frac{m^2}{\bar{\Lambda}^2}}\Big) \;=\; -\left(\gamma + \log\big(\sqrt{\frac{m^2}{\bar{\Lambda}^2}}\big)\right) + O\Big(\frac{m^2}{\bar{\Lambda}^2}\log\big(\frac{\bar{\Lambda}}{m}\big)\Big)
\tag{5.77}
$$

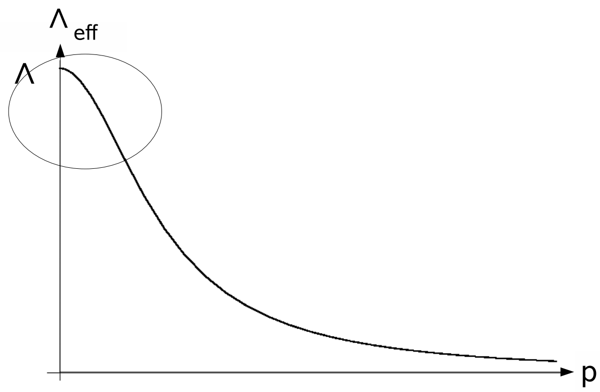

Figure 5.2: Relevant IR regime of $\Lambda_{\rm eff}(p)$

which gives

$$\begin{aligned}\Gamma_\Psi + 4\,\Gamma_\Phi &\sim \frac{g^2 n_f}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\,A_{\alpha'}(p)A_{\beta'}(-p)\,\tilde g^{\alpha'\alpha}\tilde g^{\beta'\beta}\,(p_\alpha p_\beta - \tilde g_{\alpha\beta}p\cdot p)\,\log\left(\frac{\Lambda_{\rm eff}^2}{\bar\Lambda^2}\right)\\ &= -\frac{1}{2}\,\frac{g^2 n_f}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\,\bar F_{\alpha\beta}\bar F_{\alpha'\beta'}\tilde g^{\alpha'\alpha}\tilde g^{\beta'\beta}\,\log\left(\frac{\Lambda_{\rm eff}^2}{\bar\Lambda^2}\right).\end{aligned} \tag{5.78}$$

The only approximation here is the expansion Eq. (5.77) of the Bessel functions in Eq. (5.74). Γ_Φ is the one-loop effective action for a (hermitian) scalar field as computed in [48] and reviewed in Sect. 3.3,

$$\begin{aligned}\Gamma_\Phi = -\frac{g^2}{2}\frac{1}{16\pi^2}&\int\frac{d^4p}{(2\pi)^4}\,\Big(-\frac{1}{6}\bar F_{\alpha\beta}(p)\bar F_{\alpha'\beta'}(-p)\tilde g^{\alpha'\alpha}\tilde g^{\beta'\beta}\,\log(\frac{\bar\Lambda^2}{\Lambda_{\rm eff}^2})\\ &+\frac{1}{4}(\theta\bar F(p))(\theta\bar F(-p))\Big(\Lambda_{\rm eff}^4 - \frac{1}{6}p\cdot p\,\Lambda_{\rm eff}^2 + \frac{(p\cdot p)^2}{1800}\,(47-30\log(\tfrac{p\cdot p}{\Lambda_{\rm eff}^2}))\Big)\Big).\end{aligned} \tag{5.79}$$

These expressions are valid in the IR regime of Eq. (5.76) $p\,\bar\Lambda < \Lambda_{\rm NC}^2$ corresponding to "mild" UV/IR mixing. This is the same condition which was obtained for the bosonic case [48]. We can then use the expansions

$$\Lambda_{\rm eff}^2 = \bar\Lambda^2 - p^2\frac{\bar\Lambda^4}{4\Lambda_{\rm NC}^4} + \ldots, \tag{5.80}$$

$$\Lambda_{\rm eff}^4 = \bar\Lambda^4 - p^2\frac{\bar\Lambda^6}{2\Lambda_{\rm NC}^4} + \ldots, \tag{5.81}$$

$$\log\left(\frac{\bar\Lambda^2}{\Lambda_{\rm eff}^2}\right) = \frac{1}{4}\frac{p^2\Lambda^2}{\Lambda_{\rm NC}^4} + \ldots \tag{5.82}$$

which gives

$$\begin{aligned}\Gamma_\Psi + 4\,\Gamma_\Phi &\sim \frac{1}{4}\,\frac{g^2}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\,\tilde{g}^{\alpha'\alpha}\tilde{g}^{\beta'\beta}\,\bar{F}_{\alpha\beta}(p)\bar{F}_{\alpha'\beta'}(-p)\,\frac{p^2\bar{\Lambda}^2}{\Lambda_{\rm NC}^4}\,,\\ &= \frac{1}{4}\,\frac{g^2}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\,\bar{\rho}^2\bar{\Lambda}^2p^2\,\bar{g}^{\alpha'\alpha}\bar{g}^{\beta'\beta}\,\bar{F}_{\alpha\beta}(p)\bar{F}_{\alpha'\beta'}(-p)\,,\end{aligned} \tag{5.83}$$

where $p^2 = p_\alpha p_\beta g^{\alpha\beta}$. There are obvious modifications due to the appropriate expansion of $\Lambda^2_{\rm eff}$ if one approaches the border of the IR regime as stated in Eq. (5.76).

To compare this to the geometrical results, we must take into account the different regularizations used in the heat-kernel expansion (4.30) and in the above one-loop computation. It was shown in [48] that these regularizations agree if we replace $\bar{\Lambda}^2$ with $2\bar{\Lambda}^2$ in the one-loop computation above[3]. We then find complete agreement with the result of Eq. (5.37) respectively Eq. (5.67) obtained from the geometrical point of view. Notice in particular that the induced gravitational action is nontrivial even in the case of e.g. $\mathcal{N} = 1$ supersymmetry. This is now understood in terms of induced gravity, and full cancellation is obtained only in the case of $\mathcal{N} = 4$ supersymmetry. This will be discussed below.

Finally, Γ_Ψ and Γ_Φ can be related directly to the geometrical induced action Eq. (5.37) in a more restricted IR regime, as in [48]. Assume first that $\bar{\Lambda} \ll \Lambda_{\rm NC}$. Then the IR regime (5.76) amounts to

$$p < \Lambda_{\rm NC}, \tag{5.84}$$

which is very reasonable range of validity for the classical gravity action. In this case,

$$\bar{\Lambda}^6\frac{p^2}{\Lambda_{\rm NC}^4} = \frac{\bar{\Lambda}^4}{\Lambda_{\rm NC}^4}\bar{\Lambda}^2p^2 \;\ll\; \bar{\Lambda}^2p^2 \;\sim\; \bar{\Lambda}^2 p\cdot p \tag{5.85}$$

so that we can replace

$$\Lambda_{\rm eff}^4 - \frac{1}{6}p\cdot p\Lambda_{\rm eff}^2 \;\sim\; \bar{\Lambda}^4 - \frac{1}{6}p\cdot p\bar{\Lambda}^2\,. \tag{5.86}$$

Then the leading contribution to Γ_Φ is

$$\begin{aligned}\Gamma_\Phi \sim& -\frac{g^2}{2}\frac{1}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\Big(\frac{\bar{\Lambda}^4}{4}(\theta\bar{F}(p))(\theta\bar{F}(-p)) - \frac{\bar{\Lambda}^2}{24}\bar{F}_{\alpha\beta}(p)\,\bar{F}_{\alpha'\beta'}(-p)\frac{p^2}{\bar{\Lambda}_{\rm NC}^4}\tilde{g}^{\alpha'\alpha}\tilde{g}^{\beta'\beta}\\ &\qquad - \frac{\bar{\Lambda}^2}{24}(\theta\bar{F}(p))(\theta\bar{F}(-p))p\cdot p \;+\; O(\log(\bar{\Lambda})) \;+\text{finite terms}\Big)\\ =& -\frac{g^2}{2}\frac{1}{16\pi^2}\int d^4x\,\Big(\frac{\bar{\Lambda}^4}{4}(\theta\bar{F})(\theta\bar{F}) - \bar{\rho}\,\frac{\bar{\Lambda}^2}{24}\Big(p^2\,\bar{F}_{\alpha\beta}\,F_{\alpha'\beta'}\bar{g}^{\alpha'\alpha}\bar{g}^{\beta'\beta} + (p_\alpha p_\beta\bar{g}^{\alpha\beta})(\theta\bar{F})(\theta\bar{F})\Big)\\ &\qquad + O(\log(\bar{\Lambda})) \;+\text{finite terms}\Big)\,,\end{aligned} \tag{5.87}$$

[3]This argument was strictly speaking established only for the bosonic case [48]. However, it should extend to the fermionic case without difficulties.

where again $p^2 = p_\alpha p_\beta g^{\alpha\beta}$. Taking into account again the appropriate replacement $\bar{\Lambda}^2 \to 2\bar{\Lambda}^2$ corresponding to the geometrical regularization in (4.29) and setting $n_f = 2$ for Dirac fermions, one finds as in [48] complete agreement between the above result for Γ_Ψ with the result (5.37) obtained from the geometrical point of view.

Assume finally that the condition $\bar{\Lambda} \ll \Lambda_{\rm NC}$ is violated, while maintaining the IR regime of Eq. (5.76). Then there are additional terms in the effective action Γ_Φ induced by scalars beyond Eq. (5.87), as discussed in [48]. Those correspond to noncommutative corrections beyond the semi-classical geometrical terms in Eq. (5.14).

5.4 Cancellations and supersymmetry

We compare the fermionic contribution to the gravitational action to the bosonic contribution. As it is well-known [82, 81], we note that the fermionic contribution Γ_Ψ to the one-loop effective action in NC gauge theory does not quite cancel the scalar contribution Γ_Φ, due to Eq. (5.78). From the geometrical point of view the difference corresponds to

$$\Gamma_\Psi + 4\Gamma_\Phi = \frac{1}{16\pi^2}\int \mathrm{d}^4x\, \mathrm{tr}\mathcal{E}\, \Lambda^2 = \frac{1}{16\pi^2}\int \mathrm{d}^4x\, \Big(-2\, R[\widetilde{G}] + G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma)\Big)\, \Lambda^2, \tag{5.88}$$

so that the cutoff Λ^2 should be interpreted as effective gravitational constant $1/G$. This is completely analogous to the commutative case, where the gravitational term

$$\mathrm{tr}\mathcal{E}_{\rm comm} = -R \tag{5.89}$$

is induced. The remaining UV/IR mixing cancels only in the case of $\mathcal{N} = 4$ supersymmetry,

$$\Gamma_A = -2\Gamma_\Psi - 6\Gamma_\Phi. \tag{5.90}$$

We can therefore identify Λ as the scale of $\mathcal{N} = 4$ supersymmetry breaking. Above this scale the model is assumed to be finite since $\mathcal{N} = 4$ supersymmetry is conjectured to be finite. These observations suggest that for the model to be well-defined at the quantum level, $\mathcal{N} = 4$ supersymmetry is required above the gravity scale i.e. the Planck scale. This is realized by the IKKT model [55] on a noncommutative background, see Sect. 3.4. The term $\int \mathrm{d}^4x\, \mathrm{tr}\mathcal{E}\Lambda^2$ must thus cancel with contributions coming from Γ_A. This should be checked explicitely.

Chapter 6

Conclusions

Summary. In this work we have given a rather detailed introduction to noncommutative emergent gravity. This was necessary due to the novelty of most of the results summarized in Chapter 3. The main message is that that noncommutative gauge theory does already contain gravity. There is no need to add anything. We summarize briefly the most important features of noncommutative emergent gravity.

- Spacetime is assumed to be a Poisson manifold $(\mathcal{M}, \theta^{\mu\nu}(x))$.

- Physics at the Planck scale is described by the matrix model given in Eq. (3.1) where the matrices X^a are considered to be quantized spacetime coordinates x^a.

- Physical spacetime solutions fulfill in the semi-classical limit the equations of motion, Eq. (3.46). Hence spacetime is dynamical, it is not fixed. These equations of motion play the analogue rôle of the Einstein equations in general relativity.

- Extra dimensions are implemented by scalar fields $\phi^i(X^\mu)$ which define the embedding of the $2n-1$-brane to $\mathbb{R}^D$.

- The effective metric responsible for the gravitational coupling $\widetilde{G}^{\mu\nu}(x)$ is not a fundamental object of the model. It is composed by the Poisson structure $\theta^{\mu\nu}(x)$ and the induced metric $g_{\mu\nu}(x)$. All types of fields couple to this effective metric.

- The Einstein-Hilbert action emerges at the one-loop level of perturbation theory.

- The gravitational coupling can be rewritten in terms of a noncommutative gauge coupling of a $U(1)$ gauge theory. This gives UV/IR mixing in noncommutative gauge theory a new interpretation in terms of a gravity effect.

- The theory is conjectured to be finite above some scale Λ where $\mathcal{N}=4$ supersymmetry is expected to act. This should be provided by the IKKT model in ten dimensions which is related to $\mathcal{N}=4$ super Yang-Mills theory.

The purpose of this work was to study the coupling of fermions to noncommutative emergent gravity. The issue was clarified first for the general case of branes embedded in higher dimension in Chapter 4. The reduction to the four-dimensional case and the relation to UV/IR mixing was investigated thereafter in Chapter 5. The latter was also an important check to have a least two different technial ways of evaluating the results for a flat background metric $g_{\mu\nu}$.

Fermions were coupled to the matrix model in a specific way as prescribed by the IKKT model. It is worthwhile to mention that no other coupling seems near at hand or "natural" from the matrix point of view. This matrix model action for fermions leads then in the semi-classical limit to a coupling of fermions to the geometry determined by a nontrivial effective metric

$$\widetilde{G}^{\mu\nu}_{(\tau)} = e^{-\tau}\theta^{\mu\alpha}\theta^{\nu\beta}g_{\alpha\beta}. \tag{6.1}$$

As usual in this framework this coupling is given in matrix coordinates associated with the matrix model. It was found that the spin connection vanishes in these preferred coordinates. This feature of the model is responsible for the difference with respect to the standard case.

In the case of extra dimensions, one finds that the vielbein is not simply given by the Poisson tensor $\theta^{\mu\nu}$, rather the Poisson tensor corresponds to a vielbein which relates the effective metric to the "tangential" embedding metric which in turn is non-trivial.

The main part of Chapter 4 was devoted to compute the one-loop effective action. This issue is nontrivial due to the vanishing spin connection. Known results in the literature cannot be applied. Thus a priori it was not clear whether the standard geometrical action - the Einstein-Hilbert action - would be induced. Despite of this unusual feature we have shown that the resulting fermionic action is very reasonable.

The computation of the one-loop induced action is rather complicated from a technical point of view, although in principal straight forward. In order to overcome these difficulties we proved in a first step that all induced quantities can be written in covariant form. As a consequence we were allowed to go to a normal embedding coordinate system which led to manageable expressions. However, even in this coordinate system we had to restric ourselves to two special cases:

- On-shell geometries which fulfill the equations of motion.
- Self-dual geometries which fulfill $\widetilde{G}^{\mu\nu} = g^{\mu\nu}$.

We found that the correct Einstein-Hilbert term is indeed induced at one-loop. However, there are three additional terms: One playing the rôle of a dilaton, a second term which depends on the extrinsic geometry which vanishes for on-shell geometries, and a third term which is of the form $R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$.

The second part of this work was devoted to the identification of UV/IR mixing in the geometrical picture. We showed that UV/IR mixing can be explained precisely by the gravitational point of view as expected. This provides a generalization of the results in [48] where scalar fields were treated. We have also explained why some UV/IR mixing remains even in the supersymmetric case. UV/IR mixing vanishes only for $\mathcal{N} = 4$ super Yang-Mills theory which is conjectured to be finite. Thus the cutoff introduced in the computation of the one-loop effective action should be related to the scale of $\mathcal{N} = 4$ supersymmetry breaking. These considerations suggest that the IKKT model on a noncommutative background provides a strong candidate for a consistent theory of emergent gravity which could be realized in nature.

Outlook. The framework of noncommutative emergent gravity was developed only recently. Naturally, there are many open issues.

- When it comes to fermions, the term $R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$ calls for our attention. As discussed in Sect. 4.5 there are two possible solutions so far. Either the brane tension is the dominant term and the Einstein-Hilbert term as well as the $R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$ term would only correspond to minor corrections. Secondly, the breaking of $\mathcal{N} = 4$ supersymmetry is such that the cancellation of the undesired term remains valid even at lower scales. It should be clarified if any of these possibilities is realistic.

- The term $R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$ is interesting in its own sake. It might be worthwhile to study the physics connected to it.

- The rotation of the spin in this model might provide a measurable signature for or against noncommutative emergent gravity. This should be investigated in detail such that quantitative estimates can be made.

- One of the most urgent issues is realistic solutions of (spherical) mass distributions. A first investigation was already carried out in [54]. However, this solution is not fully satisfactory since the g_{rr}-component of the metric comes with an unusual factor 1/3. It might be necessary to refine the embedding.

- The cosmological solution needs further studies. First of all, one should couple matter to the model and determine the corresponding equations of motion. Of crucial importance is a thorough investigation of the early universe in this model. Especially, the density fluctuations in the early universe should be worked out since they can be compared with observations from satellites such as WMAP and, in the future, Planck.

- So far the focus was on the "harmonic branch" of solutions, i.e. solutions where the brane tension is the dominant term. Also the "Einstein branch" deserves some attention.

- In this work only leading order contributions have been considered. Ultimately, one should go beyond the semi-classical level and investigate its corrections.

- Last but certainly not least, the finiteness of $\mathcal{N} = 4$ supersymmetry should be clarified as well as possible and realistic breaking mechanisms of this symmetry.

Appendix A

Evaluation of tr$\mathcal{E}$

We want to express tr$\mathcal{E}$,

$$\begin{aligned}\mathrm{tr}\mathcal{E} &= -\mathrm{tr}\left\{\widetilde{G}^{\mu\nu}\Omega_\mu\Omega_\nu + \widetilde{G}^{\mu\nu}\partial_\mu\Omega_\nu - \widetilde{\Gamma}^\rho\Omega_\rho\right\} \\ &= -\mathrm{tr}\Big(\frac{1}{4}\widetilde{G}_{\mu\nu}a^\mu a^\nu - \frac{1}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^\mu\widetilde{\Gamma}^\nu + \frac{1}{2}\widetilde{G}^{\mu\nu}\partial_\mu\big(\widetilde{G}_{\nu\rho}\widetilde{a}^\rho + \widetilde{G}_{\nu\rho}\widetilde{\Gamma}\rho\big)\Big).\end{aligned} \tag{A.1}$$

explicitly by the Poisson tensor $\theta^{\mu\nu}$ and the background metric $g_{\mu\nu}$.

To begin with we show that the following relation containing second order partial derivatives is true.

$$\begin{aligned}g^{\lambda\nu}G^{\rho\mu}(\partial_\rho\partial_\lambda\phi^i)(\partial_\mu\partial_\nu\phi^j)\delta_{ij} = \frac{1}{2}\Big(&g^{\rho\sigma}G^{\mu\nu}\partial_\mu\partial_\nu g_{\rho\sigma} + G^{\rho\lambda}g^{\mu\nu}\partial_\mu\partial_\nu g_{\rho\lambda} - 2G^{\rho\mu}g^{\lambda\nu}\partial_\rho\partial_\lambda g_{\mu\nu}\Big) \\ &+ e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\mu(\partial_\mu\phi^j)\right)\delta_{ij}.\end{aligned} \tag{A.2}$$

This can be seen by taking

$$\begin{aligned}(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij} + (\partial_\beta\phi^i)(\partial_\rho\partial_\sigma\partial_\delta\phi^j)\delta_{ij} = \frac{1}{2}\big(&\partial_\rho\partial_\sigma g_{\beta\delta} + \partial_\rho\partial_\delta g_{\beta\sigma} \\ &- \partial_\rho\partial_\beta g_{\sigma\delta}\big)\end{aligned} \tag{A.3}$$

and subtracting from this equation the same equation with the indices ρ and δ interchanged. This gives

$$\begin{aligned}(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij} - (\partial_\delta\phi^i)(\partial_\sigma\partial_\rho\phi^j)\delta_{ij} = \frac{1}{2}\big(&\partial_\rho\partial_\sigma g_{\beta\delta} - \partial_\rho\partial_\beta g_{\sigma\beta} \\ &- \partial_\delta\partial_\sigma g_{\beta\rho} + \partial_\delta\partial_\beta g_{\rho\sigma}\big).\end{aligned} \tag{A.4}$$

Hence we have

$$\begin{aligned}G^{\rho\sigma}g^{\beta\delta}(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij} - g^{\beta\delta}(\partial_\beta\partial_\delta\phi^i)G^{\rho\sigma}(\partial_\rho\partial_\sigma\phi^j)\delta_{ij} &= \\ G^{\rho\sigma}g^{\beta\delta}(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij} - e^\sigma g^{\beta\delta}(\partial_\beta\partial_\delta\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\mu(\partial_\mu\phi^j)\right)\delta_{ij} &= \\ \frac{1}{2}\big(G^{\mu\nu}(g\partial_\mu\partial_\nu g^{-1}) - 2G^{\rho\sigma}g^{\delta\beta}\partial_\rho\partial_\beta g_{\sigma\delta} + g^{\mu\nu}G^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}\big).&\end{aligned} \tag{A.5}$$

A simple relation is also

$$(\partial_\nu\phi^i)(\partial_\mu\partial_\lambda\phi^j)\delta_{ij}\theta^{\mu\nu} = (\partial_\mu g_{\nu\lambda})\theta^{\mu\nu}. \tag{A.6}$$

Computation of tr $(G_{\mu\nu}a^\mu a^\nu)$**.**

$$\begin{aligned}
\mathrm{tr}\,(G_{\mu\nu}a^\mu a^\nu) &= \mathrm{tr}\left[\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta} + \widetilde{\gamma}_\alpha(\partial_\rho\widetilde{\gamma}_\beta)\theta^{\rho\alpha}\theta^{\mu\beta}\right] \times \\
&\quad \left[\widetilde{\gamma}_\gamma\widetilde{\gamma}_\delta\theta^{\sigma\gamma}(\partial_\sigma\theta^{\nu\delta}) + \widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\sigma\gamma}\theta^{\nu\delta}\right] G_{\mu\nu} \\
&= \mathrm{tr}\Big[\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma\widetilde{\gamma}_\delta\theta^{\rho\alpha}\theta^{\sigma\gamma}(\partial_\rho\theta^{\mu\beta})(\partial_\sigma\theta^{\nu\delta})G_{\mu\nu} \\
&\quad + 2\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} \\
&\quad + \widetilde{\gamma}_\alpha(\partial_\rho\widetilde{\gamma}_\beta)\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}\Big]
\end{aligned} \tag{A.7}$$

We evaluate the trace of the Gamma matrices $\widetilde{\gamma}$ that appear in the above expression,

$$\begin{aligned}
\mathrm{tr}\widetilde{\gamma}_\rho\widetilde{\gamma}_\sigma\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta &= k\left(g_{\rho\sigma}g_{\alpha\beta} - g_{\rho\alpha}g_{\sigma\beta} + g_{\rho\beta}g_{\sigma\alpha}\right), \\
\mathrm{tr}\widetilde{\gamma}_\rho\gamma_{3+j}\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta &= k\left((\partial_\rho\phi^i)g_{\alpha\beta} - (\partial_\beta\phi^i)g_{\rho\alpha} + (\partial_\alpha\phi^i)g_{\rho\beta}\right)\delta_{ij}, \\
\mathrm{tr}\widetilde{\gamma}_\rho\widetilde{\gamma}_\sigma\widetilde{\gamma}_\alpha\gamma_{3+i} &= k\left((\partial_\alpha\phi^j)g_{\rho\sigma} - (\partial_\sigma\phi^j)g_{\alpha\rho} + (\partial_\rho\phi^j)g_{\alpha\sigma}\right)\delta_{ij}, \\
\mathrm{tr}\widetilde{\gamma}_\rho\gamma_{3+j}\widetilde{\gamma}_\alpha\gamma_{3+i} &= k\left(-\delta_{ij}g_{\rho\alpha} + (\delta_{kj}\delta_{li} + \delta_{ki}\delta_{jl})(\partial_\rho\phi^k)(\partial_\alpha\phi^l)\right).
\end{aligned} \tag{A.8}$$

Here k is the rank of the representation of the γ-matrices, depending on the number of extra dimensions.

$$\begin{aligned}
\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma\widetilde{\gamma}_\delta\theta^{\rho\alpha}\theta^{\sigma\gamma}(\partial_\rho\theta^{\mu\beta})(\partial_\sigma\theta^{\nu\delta})G_{\mu\nu} &= k\left(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}\right)\times \\
&\quad \theta^{\rho\alpha}\theta^{\sigma\gamma}(\partial_\rho\theta^{\mu\beta})(\partial_\sigma\theta^{\nu\delta})G_{\mu\nu} \\
&= k\Big\{ G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&\quad - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&\quad + G^{\rho\mu}G^{\sigma\nu}(\partial_\rho\theta^{-1}_{\nu\alpha})(\partial_\sigma\theta^{-1}_{\mu\beta})g^{\alpha\beta}\Big\}
\end{aligned} \tag{A.9}$$

$$
\begin{aligned}
\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} &= \mathrm{tr}\big[\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu}\big] \\
&= k\delta_{ij}\big[(\partial_\alpha\phi^j)g_{\beta\gamma} - (\partial_\beta\phi^j)g_{\alpha\gamma} + (\partial_\gamma\phi^j)g_{\alpha\beta}\big]\times \\
&\quad (\partial_\sigma\partial_\delta\phi^i)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\beta}G_{\mu\nu} \\
&= \frac{k}{2}\,\big[(\partial_\sigma g_{\alpha\delta} + \partial_\delta g_{\alpha\sigma} - \partial_\alpha g_{\sigma\delta})g_{\beta\gamma}\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} \\
&\quad - (\partial_\sigma g_{\beta\delta} + \partial_\delta g_{\beta\sigma} - \partial_\beta g_{\sigma\delta})G^{\rho\sigma}(\partial_\rho\theta^{\mu\beta})\theta^{\nu\delta}G_{\mu\nu} \\
&\quad + (\partial_\sigma g_{\gamma\delta} + \partial_\delta g_{\gamma\sigma} - \partial_\gamma g_{\sigma\delta})g_{\alpha\beta}\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu}\big] \\
&= \frac{k}{2}\,\big[G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\sigma g_{\alpha\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\delta g_{\alpha\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} \\
&\quad - G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\alpha g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\sigma g_{\beta\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} \\
&\quad - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\delta g_{\beta\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\rho\sigma}\theta^{\mu\beta}(\partial_\beta g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} \\
&\quad - 2G^{\rho\mu}(\partial_\rho\theta^{-1}_{\mu\nu})G^{\sigma\lambda}(\partial_\sigma\theta^{-1}_{\lambda\delta})g^{\nu\delta}\big]
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
\widetilde{\gamma}_\alpha(\partial_\rho\widetilde{\gamma}_\beta)\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}\widetilde{\gamma}^{\sigma\gamma}g^{\beta\delta} &= \mathrm{tr}\big[\widetilde{\gamma}_\alpha(\partial_\rho\widetilde{\gamma}_\beta)\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}\big] \\
&= k\,\big[-\delta_{ij}g_{\alpha\gamma} + (\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li})(\partial_\alpha\phi^k)(\partial_\gamma\phi^l)\big]\times \\
&\quad (\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta} \\
&= k\,\big[-(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij}G^{\rho\sigma}g^{\beta\delta} + (\partial_\rho g_{\alpha\beta})\theta^{\rho\alpha}(\partial_\sigma g_{\gamma\delta})\theta^{\sigma\gamma}g^{\beta\delta} \\
&\quad + \frac{1}{4}(\partial_\sigma g_{\alpha\delta} + \partial_\delta g_{\alpha\sigma} - \partial_\alpha g_{\sigma\delta})(\partial_\rho g_{\gamma\beta} + \partial_\beta g_{\gamma\rho} - \partial_\gamma g_{\rho\beta})\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}\big] \\
&= k\,\big[-\frac{1}{2}\big(G^{\mu\nu}(g\partial_\mu\partial_\nu g^{-1}) - 2G^{\rho\sigma}g^{\delta\beta}\partial_\rho\partial_\beta g_{\sigma\delta} + g^{\mu\nu}G^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}\big) \\
&\quad + G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&\quad + \frac{1}{2}\theta^{\rho\alpha}(\partial_\rho g_{\gamma\beta})\theta^{\sigma\gamma}(\partial_\sigma g_{\alpha\delta})g^{\delta\beta} \\
&\quad + \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\mu g_{\rho\alpha})(\partial_\nu g_{\sigma\beta})g^{\alpha\beta} \\
&\quad - \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\alpha g_{\mu\rho})(\partial_\beta g_{\nu\sigma})g^{\alpha\beta} \\
&\quad - e^\sigma g^{\beta\delta}(\partial_\beta\partial_\delta\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^{\mu\nu}(\partial_\mu\phi^j)\right)\delta_{ij}\big]
\end{aligned}
\tag{A.11}
$$

In the last step we have used Eq. (A.5). The explicit expression for the whole term is then

$$
\begin{aligned}
-\frac{e^{-\sigma}}{4}\mathrm{tr}\,(G^{\mu\nu}a_\mu a_\nu) = -k\frac{e^{-\sigma}}{4}\Big\{ & G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
& - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + G^{\rho\mu}G^{\sigma\nu}(\partial_\rho\theta^{-1}_{\nu\alpha})(\partial_\sigma\theta^{-1}_{\mu\beta})g^{\alpha\beta} \\
& + G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\sigma g_{\alpha\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\delta g_{\alpha\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} \\
& - G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\alpha g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\sigma g_{\beta\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} \\
& - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\delta g_{\beta\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\rho\sigma}\theta^{\mu\beta}(\partial_\beta g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} \\
& - 2G^{\rho\mu}(\partial_\rho\theta^{-1}_{\mu\nu})G^{\sigma\lambda}(\partial_\sigma\theta^{-1}_{\lambda\delta})g^{\nu\delta} \\
& - \frac{1}{2}\big(G^{\mu\nu}(g\partial_\mu\partial_\nu g^{-1}) - 2G^{\rho\sigma}g^{\delta\beta}\partial_\rho\partial_\beta g_{\sigma\delta} + g^{\mu\nu}G^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}\big) \\
& + G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + \frac{1}{2}\theta^{\rho\alpha}(\partial_\rho g_{\gamma\beta})\theta^{\sigma\gamma}(\partial_\sigma g_{\alpha\delta})g^{\delta\beta} \\
& + \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\mu g_{\rho\alpha})(\partial_\nu g_{\sigma\beta})g^{\alpha\beta} - \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\alpha g_{\mu\rho})(\partial_\beta g_{\nu\sigma})g^{\alpha\beta}\Big\} \\
& + \frac{k}{4}g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\mu\partial_\mu\phi^j\right)\delta_{ij}.
\end{aligned}
\tag{A.12}
$$

Computation of tr$(-G^{\mu\nu}(\partial_\mu G_{\nu\rho})a^\rho)$ and tr$(\partial_\mu a^\mu)$. Next we deal with the remaining two term in tr$\mathcal{E}$. Both turn out to be zero for on-shell geometries. We evaluate again the trace.

$$
\begin{aligned}
\mathrm{tr}(\partial_\mu\widetilde{\gamma}_\alpha)\widetilde{\gamma}_\beta &= \mathrm{tr}\big[\gamma_{3+i}(\partial_\mu\partial_\alpha\phi^i)(\gamma_\beta + \gamma_{3+j}(\partial_\beta\phi^j))\big] \\
&= k\delta_{ij}(\partial_\mu\partial_\alpha\phi^i)(\partial_\beta\phi^j) \\
&= \frac{k}{2}(\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\mu\beta} - \partial_\beta g_{\mu\alpha})
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
\mathrm{tr}(\partial_\mu\widetilde{\gamma}_\alpha)(\partial_\nu\widetilde{\gamma}_\beta)\theta^{\nu\alpha}\theta^{\mu\beta} &= \mathrm{Tr}\gamma_{3+i}\gamma_{3+j}(\partial_\mu\partial_\alpha\phi^i)(\partial_\nu\partial_\beta\phi^j)\theta^{\nu\alpha}\theta^{\mu\beta} \\
&= k\delta_{ij}(\partial_\mu\partial_\alpha\phi^i)(\partial_\nu\partial_\beta\phi^j)\theta^{\nu\alpha}\theta^{\mu\beta} \\
&= \frac{1}{2}(\partial_\mu\partial_\nu g_{\alpha\beta} + \partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\mu\partial_\alpha g_{\nu\beta})\theta^{\nu\alpha}\theta^{\mu\beta} \\
&= \theta^{\mu\alpha}\theta^{\nu\beta}\partial_\mu\partial_\nu g_{\alpha\beta}
\end{aligned}
\tag{A.14}
$$

First we consider the computation of $-G^{\mu\nu}(\partial_\mu G_{\nu\rho})a^\rho$.

$$
\begin{aligned}
\mathrm{tr}a^\rho &= \mathrm{Tr}\big[\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\theta^{\sigma\alpha}(\partial_\sigma\theta^{\rho\beta}) + \widetilde{\gamma}_\alpha(\partial_\sigma\widetilde{\gamma}_\beta)\theta^{\sigma\alpha}\theta^{\rho\beta}\big] \\
&= k\big[g_{\alpha\beta}\theta^{\sigma\alpha}(\partial_\sigma\theta^{\rho\beta}) + \frac{1}{2}(\partial_\sigma g_{\alpha\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\alpha g_{\sigma\beta})\theta^{\sigma\alpha}\theta^{\rho\beta}\big] \\
&= k\big[\theta^{\sigma\alpha}(\partial_\sigma\theta^{\rho\beta})g_{\alpha\beta} + \theta^{\sigma\alpha}\theta^{\rho\alpha}(\partial_\sigma g_{\alpha\beta})\big] \\
&= -k\,e^\sigma\widetilde{\Gamma}^\rho.
\end{aligned}
\tag{A.15}
$$

The remaining term $\mathrm{tr}\partial_\mu a^\mu$ gives

$$\begin{aligned}
\mathrm{tr}\partial_\mu a^\mu &= \mathrm{Tr}\big[(\partial_\mu\widetilde{\gamma}_\alpha)\widetilde{\gamma}_\beta\theta^{\nu\alpha}(\partial_\nu\theta^{\mu\beta}) + \widetilde{\gamma}_\alpha(\partial_\mu\widetilde{\gamma}_\beta)\theta^{\nu\alpha}(\partial_\nu\theta^{\mu\beta}) \\
&\quad + \widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta}) + \widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\theta^{\nu\beta}\partial_\nu\partial_\mu\theta^{\mu\beta} \\
&\quad + (\partial_\mu\widetilde{\gamma}_\alpha)(\partial_\nu\widetilde{\gamma}_\beta)\theta^{\nu\alpha}\theta^{\mu\beta} + \widetilde{\gamma}_\alpha(\partial_\nu\widetilde{\gamma}_\beta)(\partial_\mu\theta^{\nu\alpha})\theta^{\mu\beta} \\
&\quad + \widetilde{\gamma}_\alpha(\partial_\nu\widetilde{\gamma}_\beta)\theta^{\nu\alpha}(\partial_\mu\theta^{\mu\beta})\big] \\
&= \frac{k}{2}\big\{(\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\mu\beta} - \partial_\beta g_{\mu\alpha})\theta^{\nu\beta}(\partial_\nu\theta^{\mu\beta}) \\
&\quad + (\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\mu\alpha} - \partial_\alpha g_{\mu\beta})\theta^{\nu\beta}(\partial_\nu\theta^{\mu\beta}) \\
&\quad + 2(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} + 2\theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + 2\theta^{\mu\alpha}\theta^{\nu\beta}\partial_\mu\partial_\nu g_{\alpha\beta} \\
&\quad + (\partial_\nu g_{\alpha\beta} + \partial_\beta g_{\alpha\nu} - \partial_\alpha g_{\nu\beta})(\partial_\mu\theta^{\nu\alpha})\theta^{\mu\beta} \\
&\quad + (\partial_\nu g_{\alpha\beta} + \partial_\beta g_{\alpha\nu} - \partial_\alpha g_{\nu\beta})\theta^{\nu\alpha}(\partial_\mu\theta^{\mu\beta})\big\} \\
&= k\,\big\{2\theta^{\mu\alpha}(\partial_\mu\theta^{\nu\beta})(\partial_\nu g_{\alpha\beta}) + \theta^{\mu\alpha}(\partial_\nu\theta^{\nu\beta})(\partial_\mu g_{\alpha\beta}) \\
&\quad + (\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} + \theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + \theta^{\mu\alpha}\theta^{\nu\beta}\partial_\mu\partial_\nu g_{\alpha\beta}\big\} \\
&= k\,\partial_\mu\big\{\theta^{\nu\alpha}(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} + \theta^{\nu\alpha}\theta^{\mu\beta}(\partial_\nu g_{\alpha\beta})\big\} \\
&= -k\,\partial_\mu\left(e^\sigma\widetilde{\Gamma}^\mu\right).
\end{aligned} \tag{A.16}$$

Due to Eq. (A.15) and (A.16) we find

$$\mathrm{tr}\left(\widetilde{G}^{\mu\nu}\partial_\mu\big(\widetilde{G}_{\nu\rho}\widetilde{a}^\rho + \widetilde{G}_{\nu\rho}\widetilde{\Gamma}^\rho\big)\right) = 0 \tag{A.17}$$

That means also that for on-shell geometries $\mathrm{tr}\mathcal{E}$ is given solely by

$$\mathrm{tr}\mathcal{E} = -\frac{e^{-\sigma}}{4}\mathrm{tr}\,G^{\mu\nu}a_\mu a_\nu. \tag{A.18}$$

For general geometries we have then

$$
\begin{aligned}
\mathrm{tr}\mathcal{E} &= -\mathrm{tr}\left(\frac{1}{4}\widetilde{G}_{\mu\nu}\widetilde{a}^{\mu}\widetilde{a}^{\nu} - \frac{1}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^{\mu}\widetilde{\Gamma}^{\nu}\right)\\
&= -k\frac{e^{-\sigma}}{4}\Big\{G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta}\\
&\quad - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + G^{\rho\mu}G^{\sigma\nu}(\partial_\rho\theta^{-1}_{\nu\alpha})(\partial_\sigma\theta^{-1}_{\mu\beta})g^{\alpha\beta}\\
&\quad + G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\sigma g_{\alpha\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\delta g_{\alpha\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta}\\
&\quad - G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\alpha g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\sigma g_{\beta\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta}\\
&\quad - G^{\rho\sigma}\theta^{\mu\beta}(\partial_\delta g_{\beta\sigma})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta} + G^{\rho\sigma}\theta^{\mu\beta}(\partial_\beta g_{\sigma\delta})(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta}\\
&\quad - 2G^{\rho\mu}(\partial_\rho\theta^{-1}_{\mu\nu})G^{\sigma\lambda}(\partial_\sigma\theta^{-1}_{\lambda\delta})g^{\nu\delta}\\
&\quad - \frac{1}{2}\big(G^{\mu\nu}(g\partial_\mu\partial_\nu g^{-1}) - 2G^{\rho\sigma}g^{\delta\beta}\partial_\rho\partial_\beta g_{\sigma\delta} + g^{\mu\nu}G^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}\big)\\
&\quad + G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + \frac{1}{2}\theta^{\rho\alpha}(\partial_\rho g_{\gamma\beta})\theta^{\sigma\gamma}(\partial_\sigma g_{\alpha\delta})g^{\delta\beta}\\
&\quad + \frac{1}{2}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\mu g_{\rho\alpha})(\partial_\nu g_{\sigma\beta})g^{\alpha\beta} - \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\alpha g_{\mu\rho})(\partial_\beta g_{\nu\sigma})g^{\alpha\beta}\Big\}\\
&\quad + \frac{k}{4}g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^{\mu}\partial_\mu\phi^j\right)\delta_{ij}\\
&\quad + \frac{k}{4}\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^{\mu}\widetilde{\Gamma}^{\nu}.
\end{aligned}
\tag{A.19}
$$

Appendix B

Covariance of tr$\mathcal{E}$

We aim to show that tr$\mathcal{E}$ can be written in covariant manner. If so, we can change to a normal coordinate system, which will simplify tr$\mathcal{E}$ and the Ricci scalar enormously. However, notice that now tr$\mathcal{E}$ should be related to the Ricci scalar directly and not only under the integral, where we would be allowed to use partial integration. Since normal coordinates make sense only at a point, partial integration is not admissible here.

Notation. We distinguish between the *effective metric* $\widetilde{G}_{\mu\nu}$ and the *background metric* $g_{\mu\nu}$. The covariant derivatives and Christoffel symbols with respect to the background metric $g_{\mu\nu}$ as important in this section are written as ∇_μ and $\Gamma^{\mu}_{\rho\sigma}$.

By using expressions containing derivatives of $g_{\mu\nu}$,

$$\begin{aligned}
\partial_\lambda g_{\mu\nu} &= (\partial_\lambda \partial_\mu \phi^i)(\partial_\nu \phi^j)\delta_{ij} + (\partial_\mu \phi^i)(\partial_\lambda \partial_\nu \phi^j)\delta_{ij}, \\
\partial_\nu g_{\lambda\mu} &= (\partial_\nu \partial_\lambda \phi^i)(\partial_\mu \phi^j)\delta_{ij} + (\partial_\lambda \phi^i)(\partial_\nu \partial_\mu \phi^j)\delta_{ij}, \\
\partial_\mu g_{\nu\lambda} &= (\partial_\mu \partial_\nu \phi^i)(\partial_\lambda \phi^j)\delta_{ij} + (\partial_\nu \phi^i)(\partial_\mu \partial_\lambda \phi^j)\delta_{ij},
\end{aligned} \tag{B.1}$$

we find the following relation

$$\begin{aligned}
(\partial_\lambda \phi^i)(\partial_\nu \partial_\mu \phi^j)\delta_{ij} &= \frac{1}{2}\Big(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\Big) \\
&= \Gamma^{\sigma}_{\mu\nu} g_{\sigma\lambda}.
\end{aligned} \tag{B.2}$$

Now we are able to rewrite tr$\mathcal{E}$ in terms of the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}[g]$.

$$
\begin{aligned}
\mathrm{tr}\mathcal{E} &= -\frac{e^{-\sigma}}{4}\,\mathrm{tr}\,(a^\mu a^\nu G_{\mu\nu}) + \frac{1}{4}\,\mathrm{tr}\left(\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^\mu\widetilde{\Gamma}^\nu\right) \\
&= -\frac{e^{-\sigma}}{4}\,\mathrm{tr}\Big[\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma\widetilde{\gamma}_\delta\theta^{\rho\alpha}\theta^{\sigma\gamma}(\partial_\rho\theta^{\mu\beta})(\partial_\sigma\theta^{\nu\delta})G_{\mu\nu} \\
&\quad + 2\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} \\
&\quad + \widetilde{\gamma}_\alpha(\partial_\rho\widetilde{\gamma}_\beta)\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}\Big] \\
&\quad + \frac{e^{-\sigma}}{4}\,\mathrm{tr}\left(\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^\mu\widetilde{\Gamma}^\nu\right) \\
&= -\frac{e^{-\sigma}}{4}\,k\,\Big[(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma})\theta^{\rho\alpha}\theta^{\sigma\gamma}(\partial_\rho\theta^{\mu\beta})(\partial_\sigma\theta^{\nu\delta})G_{\mu\nu} \\
&\quad + 2\delta_{ij}\big[(\partial_\alpha\phi^j)g_{\beta\gamma} - (\partial_\beta\phi^j)g_{\alpha\gamma} + (\partial_\gamma\phi^j)g_{\alpha\beta}\big](\partial_\sigma\partial_\delta\phi^i)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} \\
&\quad + \big[-\delta_{ij}g_{\alpha\gamma} + (\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li})(\partial_\alpha\phi^k)(\partial_\gamma\phi^l)\big](\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}\Big] \\
&\quad + \frac{1}{4}\,\mathrm{tr}\left(\widetilde{G}_{\mu\nu}\widetilde{\Gamma}^\mu\widetilde{\Gamma}^\nu\right).
\end{aligned}
\tag{B.3}
$$

Since the first term in Eq. (B.3) does not contain a partial derivative of $g_{\mu\nu}$, we begin with the second term,

$$
\begin{aligned}
2\,\mathrm{tr}\left(\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\rho)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu}\right) &= 2\,k\,\delta_{ij}\Big((\partial_\alpha\phi^i)g_{\beta\gamma} - (\partial_\beta\phi^j)g_{\alpha\gamma} + (\partial_\gamma\phi^j)g_{\alpha\beta}\Big)\times \\
&\quad (\partial_\sigma\partial_\delta\phi^i)\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} \\
&= 2\,k\,\Big(g_{\alpha\lambda}\Gamma^\lambda_{\sigma\delta}g_{\beta\delta}\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} \\
&\quad - g_{\beta\lambda}\Gamma^\lambda_{\sigma\delta}g_{\alpha\gamma}\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu} \\
&\quad + g_{\gamma\lambda}\Gamma^\lambda_{\sigma\delta}g_{\alpha\beta}\theta^{\rho\alpha}(\partial_\rho\theta^{\mu\beta})\theta^{\sigma\gamma}\theta^{\nu\delta}G_{\mu\nu}\Big) \\
&= 2\,k\,\Big(\underbrace{g_{\alpha\lambda}\Gamma^\lambda_{\sigma\delta}G^{\mu\sigma}\theta^{\rho\alpha}(\partial_\rho\theta^{-1}_{\mu\nu})g^{\nu\delta}}_{(a)} \\
&\quad - \underbrace{g_{\alpha\lambda}\Gamma^\lambda_{\sigma\delta}G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\mu\nu})\theta^{\mu\alpha}g^{\nu\delta}}_{(b)} \\
&\quad + \underbrace{g_{\alpha\lambda}\Gamma^\lambda_{\sigma\delta}G^{\rho\mu}(\partial_\rho\theta^{-1}_{\mu\nu})\theta^{\sigma\alpha}g^{\nu\delta}}_{(c)}\Big).
\end{aligned}
\tag{B.4}
$$

Next we address the third term in Eq. (B.3). We write

$$
(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij} + (\partial_\beta\phi^i)(\partial_\rho\partial_\sigma\partial_\delta\phi^j)\delta_{ij} = (\partial_\rho g_{\beta\lambda})\Gamma^\lambda_{\sigma\delta} + g_{\beta\lambda}(\partial_\rho\Gamma^\lambda_{\sigma\delta}). \tag{B.5}
$$

and subtract from this equation the same equation, interchanging this time the indices ρ and δ. This gives

$$
\begin{aligned}
&(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij}G^{\rho\sigma}g^{\delta\beta} - g^{\delta\beta}(\partial_\delta\partial_\beta\phi^i)G^{\rho\sigma}(\partial_\rho\partial_\sigma\phi^j)\delta_{ij} = \\
&(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\delta_{ij}G^{\rho\sigma}g^{\delta\beta} - e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^i + \widetilde{\Gamma}^\rho\partial_\rho\phi^j\right)\delta_{ij} = \\
&G^{\rho\sigma}g^{\delta\beta}(\partial_\rho g_{\beta\lambda})\Gamma^\lambda_{\sigma\delta} + G^{\rho\sigma}(\partial_\rho\Gamma^\lambda_{\sigma\lambda}) - G^{\rho\sigma}g^{\beta\delta}(\partial_\delta g_{\beta\lambda})\Gamma^\lambda_{\rho\sigma} - G^{\rho\sigma}(\partial_\lambda\Gamma^\lambda_{\rho\sigma}).
\end{aligned}
\tag{B.6}
$$

Using

$$
\partial_\rho g_{\beta\lambda} = \Gamma^\eta_{\rho\beta}g_{\eta\lambda} + \Gamma^\eta_{\rho\lambda}g_{\beta\eta}
\tag{B.7}
$$

one finds

$$
\begin{aligned}
(\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)G^{\rho\sigma}g^{\delta\beta} &= G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\rho\beta}\Gamma^\lambda_{\sigma\delta}g_{\eta\lambda} + G^{\rho\sigma}\Gamma^\delta_{\rho\lambda}\Gamma^\lambda_{\sigma\delta} + G^{\rho\sigma}(\partial_\rho\Gamma^\lambda_{\sigma\lambda}) \\
&\quad - G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\delta\beta}\Gamma^\lambda_{\rho\sigma}g_{\eta\lambda} - G^{\rho\sigma}\Gamma^\eta_{\eta\lambda}\Gamma^\lambda_{\rho\sigma} - G^{\rho\sigma}(\partial_\lambda\Gamma^\lambda_{\rho\sigma}) \\
&\quad + e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^i + \widetilde{\Gamma}^\rho\partial_\rho\phi^j\right)\delta_{ij} \\
&= G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\rho\beta}\Gamma^\lambda_{\sigma\delta}g_{\eta\lambda} - G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\delta\beta}\Gamma^\lambda_{\rho\sigma}g_{\eta\lambda} \\
&\quad + G^{\rho\sigma}\left\{\Gamma^\delta_{\rho\lambda}\Gamma^\lambda_{\sigma\delta} - \Gamma^\lambda_{\rho\sigma}\Gamma^\delta_{\delta\lambda} + \partial_\rho\Gamma^\lambda_{\sigma\lambda} - \partial_\lambda\Gamma^\lambda_{\rho\sigma}\right\} \\
&\quad + e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^i + \widetilde{\Gamma}^\rho\partial_\rho\phi^j\right)\delta_{ij} \\
&= G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\rho\beta}\Gamma^\lambda_{\sigma\delta}g_{\eta\lambda} - G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\delta\beta}\Gamma^\lambda_{\rho\sigma}g_{\eta\lambda} \\
&\quad + e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^i + \widetilde{\Gamma}^\rho\partial_\rho\phi^j\right)\delta_{ij} \\
&\quad - G^{\mu\nu}R_{\mu\nu}[g].
\end{aligned}
\tag{B.8}
$$

We obtain for the third term of tr$\mathcal{E}$

$$
\begin{aligned}
\mathrm{tr}\left(\widetilde{\gamma}_\alpha(\partial_\rho\widetilde{\gamma}_\beta)\widetilde{\gamma}_\gamma(\partial_\sigma\widetilde{\gamma}_\delta)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}\right) &= k\Big(-\delta_{ij}g_{\alpha\gamma} + (\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li})(\partial_\alpha\phi^k)(\partial_\gamma\phi^l)\Big)\times \\
&\quad (\partial_\rho\partial_\beta\phi^i)(\partial_\sigma\partial_\delta\phi^j)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta} \\
&= k\Big(-\underbrace{G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\rho\beta}\Gamma^\lambda_{\sigma\delta}g_{\eta\lambda}}_{(f)} + G^{\rho\sigma}g^{\delta\beta}\Gamma^\eta_{\delta\beta}\Gamma^\lambda_{\rho\sigma}g_{\eta\lambda} \\
&\quad + G^{\mu\nu}R_{\mu\nu}[g] - e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^i + \widetilde{\Gamma}^\rho\partial_\rho\phi^j\right)\delta_{ij} \\
&\quad + \underbrace{g_{\alpha\lambda}\Gamma^\lambda_{\rho\beta}g_{\gamma\lambda'}\Gamma^{\lambda'}_{\sigma\delta}\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}}_{(d)} + \underbrace{g_{\alpha\lambda}\Gamma^\lambda_{\sigma\delta}g_{\gamma\lambda'}\Gamma^{\lambda'}_{\rho\beta}\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}}_{(e)}\Big).
\end{aligned}
\tag{B.9}
$$

Let us write the result in an unconventional but simple way.

$$\begin{aligned}-\frac{e^{-\sigma}}{4}\mathrm{tr}G_{\mu\nu}a^\mu a^\nu = -e^{-\sigma}\frac{k}{4}\Big\{ & G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ & + G^{\rho\mu}G^{\sigma\nu}(\partial_\rho\theta^{-1}_{\nu\alpha})(\partial_\sigma\theta^{-1}_{\mu\beta})g^{\alpha\beta} + \sum_{i=a}^{f}(i) + G^{\mu\nu}R_{\mu\nu}[g] + g^{\mu\nu}\Gamma^\lambda_{\mu\nu}G^{\rho\sigma}\Gamma^\eta_{\rho\sigma}g_{\lambda\eta} \\ & - e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\rho(\partial_\rho\phi^j)\right)\delta_{ij}\Big\},\end{aligned} \tag{B.10}$$

where the terms $(i), i = a \dots f$ refer to the terms denoted via curly brace.

Next consider the first term of Eq. (B.3),

$$\begin{aligned}\widetilde{\gamma}_\alpha\widetilde{\gamma}_\beta\widetilde{\gamma}_\gamma\widetilde{\gamma}_\delta\theta^{\rho\alpha}\theta^{\sigma\gamma}(\partial_\rho\theta^{\mu\beta})(\partial_\sigma\theta^{\nu\delta})G_{\mu\nu} = k\Big\{ & G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ & - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ & + G^{\rho\mu}G^{\sigma\nu}(\partial_\rho\theta^{-1}_{\nu\alpha})(\partial_\sigma\theta^{-1}_{\mu\beta})g^{\alpha\beta}\Big\}.\end{aligned} \tag{B.11}$$

In order to write tr$\mathcal{E}$ covariantly we replace every partial derivative by a covariant derivative ∇_μ.

$$\begin{aligned}G^{\mu\nu}(\nabla_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\nabla_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} &= G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\nu\alpha} - \Gamma^\lambda_{\mu\nu}\theta^{-1}_{\lambda\alpha} - \Gamma^\lambda_{\mu\alpha}\theta^{-1}_{\nu\lambda})\times \\ &\quad (\partial_\rho\theta^{-1}_{\sigma\beta} - \Gamma^{\lambda'}_{\rho\sigma}\theta^{-1}_{\lambda'\beta} - \Gamma^{\lambda'}_{\rho\beta}\theta^{-1}_{\sigma\lambda'})g^{\alpha\beta} \\ &= G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\nu\alpha})(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - 2G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\nu}\theta^{-1}_{\lambda\alpha}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ &\quad + 2\Gamma^\lambda_{\mu\alpha}\theta^{\mu\nu}g_{\nu\lambda}G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\nu}\Gamma^{\lambda'}_{\rho\sigma}G_{\lambda\lambda'} \\ &\quad - 2G^{\rho\sigma}\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\rho\sigma}\theta^{\mu\nu}g_{\nu\lambda}\theta^{-1}_{\lambda'\beta}g^{\alpha\beta} + \Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\rho\beta}\theta^{\mu\nu}g_{\nu\lambda}\theta^{\rho\sigma}g_{\sigma\lambda'}g^{\alpha\beta}\end{aligned} \tag{B.12}$$

$$\begin{aligned}G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} &= G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha} - \Gamma^\lambda_{\mu\rho}\theta^{-1}_{\lambda\alpha} - \Gamma^\lambda_{\mu\alpha}\theta^{-1}_{\rho\lambda})\times \\ &\quad (\partial_\nu\theta^{-1}_{\sigma\beta} - \Gamma^{\lambda'}_{\nu\sigma}\theta^{-1}_{\lambda'\beta} - \Gamma^{\lambda'}_{\nu\beta}\theta^{-1}_{\sigma\lambda'})g^{\alpha\beta} \\ &= G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta}) - 2G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\rho}\theta^{-1}_{\lambda\alpha}(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ &\quad + 2G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\theta^{\sigma\rho}g_{\rho\lambda}(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + \Gamma^\lambda_{\mu\rho}\Gamma^{\lambda'}_{\nu\sigma}G^{\mu\nu}G^{\rho\sigma}G_{\lambda\lambda'} \\ &\quad - 2G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\nu\sigma}\theta^{\sigma\rho}g_{\rho\lambda}\theta^{-1}_{\lambda'\beta}g^{\alpha\beta} + G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\nu\beta}g^{\alpha\beta}g_{\lambda\lambda'}\end{aligned} \tag{B.13}$$

$$
\begin{aligned}
G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} &= G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha} - \Gamma^\lambda_{\mu\rho}\theta^{-1}_{\lambda\alpha} - \Gamma^\lambda_{\mu\alpha}\theta^{-1}_{\rho\lambda})\times \\
&\quad (\partial_\sigma\theta^{-1}_{\nu\beta} - \Gamma^{\lambda'}_{\sigma\nu}\theta^{-1}_{\lambda'\beta} - \Gamma^{\lambda'}_{\sigma\beta}\theta^{-1}_{\nu\lambda'})g^{\alpha\beta} \\
&= G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} - 2G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\rho}\theta^{-1}_{\lambda\alpha}(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
&\quad + 2G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\theta^{\sigma\rho}g_{\rho\lambda}(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} + G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\rho}\Gamma^{\lambda'}_{\sigma\nu}G_{\lambda\lambda'} \\
&\quad - 2G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\sigma\nu}\theta^{\sigma\rho}g_{\rho\lambda}\theta^{-1}_{\lambda'\beta}g^{\alpha\beta} + \Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\sigma\beta}\theta^{\sigma\rho}g_{\rho\lambda}\theta^{\mu\nu}g_{\nu\lambda'}g^{\alpha\beta}
\end{aligned} \tag{B.14}
$$

Combining the above three terms gives in total

$$
\begin{aligned}
& k\Big(G^{\mu\nu}(\nabla_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\nabla_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&+ G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\Big) = k\Big(G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&- G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
&- 2G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\nu}\theta^{-1}_{\lambda\alpha}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + 2\,\underbrace{\Gamma^\lambda_{\mu\alpha}\theta^{\mu\nu}g_{\nu\lambda}G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta}}_{(c)} \\
&+ G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\nu}\Gamma^{\lambda'}_{\rho\sigma}G_{\lambda\lambda'} - 2G^{\rho\sigma}\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\rho\sigma}\theta^{\mu\nu}g_{\nu\lambda}\theta^{-1}_{\lambda'\beta}g^{\alpha\beta} \\
&+ \underbrace{\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\rho\beta}\theta^{\mu\nu}g_{\nu\lambda}\theta^{\rho\sigma}g_{\sigma\lambda'}g^{\alpha\beta}}_{(d)} + 2G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\rho}\theta^{-1}_{\lambda\alpha}(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&- 2\,\underbrace{G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\theta^{\sigma\rho}g_{\rho\lambda}(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta}}_{(b)} - \underbrace{G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\nu\beta}g^{\alpha\beta}g_{\lambda\lambda'}}_{(f)} \\
&- 2G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\rho}\theta^{-1}_{\lambda\alpha}(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} + 2\,\underbrace{G^{\mu\nu}\Gamma^\lambda_{\mu\alpha}\theta^{\sigma\rho}g_{\rho\lambda}(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}}_{(a)} \\
&+ \underbrace{\Gamma^\lambda_{\mu\alpha}\Gamma^{\lambda'}_{\sigma\beta}\theta^{\sigma\rho}g_{\rho\lambda}\theta^{\mu\nu}g_{\nu\lambda'}g^{\alpha\beta}}_{(e)}\Big) \\
&= k\Big(G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&+ G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} + \sum_{i=a}^{f}(i) \\
&- 2\,e^\sigma G^{\mu\nu}\Gamma^\lambda_{\mu\nu}G_{\lambda\eta}\widetilde{\Gamma}^\eta + G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\nu}\Gamma^\eta_{\rho\sigma}G_{\lambda\eta}\Big) \\
&= \mathrm{tr}a^\mu a^\nu G_{\mu\nu} - k\,G^{\mu\nu}R_{\mu\nu}[g] - k\,g^{\mu\nu}\Gamma^\lambda_{\mu\nu}G^{\rho\sigma}\Gamma^\eta_{\rho\sigma}g_{\eta\lambda} \\
&+ k\,e^\sigma g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\Big(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^\rho(\partial_\rho\phi^j)\Big)\delta_{ij} \\
&- 2\,e^\sigma G^{\mu\nu}\Gamma^\lambda_{\mu\nu}G_{\lambda\eta}\widetilde{\Gamma}^\eta + G^{\mu\nu}G^{\rho\sigma}\Gamma^\lambda_{\mu\nu}\Gamma^\eta_{\rho\sigma}G_{\lambda\eta}.
\end{aligned} \tag{B.15}
$$

In the above equation two terms cancel due to

$$\begin{aligned} G^{\mu\nu}G^{\rho\sigma}\Gamma^{\lambda}_{\mu\rho}\theta^{-1}_{\lambda\alpha}(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} &= G^{\rho\nu}G^{\mu\sigma}\Gamma^{\lambda}_{\mu\rho}\theta^{-1}_{\lambda\alpha}(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\ &= G^{\mu\nu}G^{\rho\sigma}\Gamma^{\lambda}_{\mu\rho}\theta^{-1}_{\lambda\alpha}(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}. \end{aligned} \tag{B.16}$$

We also exploited the following relations,

$$\begin{aligned} G^{\mu\nu}G^{\rho\sigma}\Gamma^{\lambda}_{\mu\nu}\theta^{-1}_{\lambda\alpha}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} + G^{\rho\sigma}\Gamma^{\lambda}_{\mu\alpha}\Gamma^{\lambda'}_{\rho\sigma}\theta^{\mu\nu}g_{\nu\lambda}\theta^{-1}_{\lambda'\beta}g^{\alpha\beta} &= G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}\theta^{-1}_{\lambda\beta}g^{\alpha\beta}\times \\ &\quad \left(G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\alpha}) + \theta^{\rho\sigma}\Gamma^{\eta}_{\rho\alpha}g_{\sigma\eta}\right) \\ &= -G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}G_{\lambda\sigma}\times \\ &\quad \left(\theta^{\rho\alpha}(\partial_\rho\theta^{\sigma\beta})g_{\alpha\beta} + \theta^{\rho\alpha}\theta^{\sigma\beta}(\partial_\rho g_{\alpha\beta})\right) \\ &= e^{\sigma}G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}G_{\lambda\sigma}\widetilde{\Gamma}^{\sigma}, \end{aligned} \tag{B.17}$$

and

$$\theta^{\rho\sigma}(\partial_\rho g_{\sigma\beta}) = \Gamma^{\lambda}_{\rho\alpha}g_{\lambda\sigma}\theta^{\rho\sigma}. \tag{B.18}$$

In the end we obtain for $\mathrm{tr}a^\mu a^\nu G_{\mu\nu}$ the following result,

$$\begin{aligned} \mathrm{tr}\, a^\mu a^\nu G_{\mu\nu} = k\Big(& G^{\mu\nu}(\nabla_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\nabla_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ &+ G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} + G^{\mu\nu}R_{\mu\nu}[g] + g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}G^{\rho\sigma}\Gamma^{\eta}_{\rho\sigma}g_{\eta\lambda} \\ &- e^{\sigma}g^{\mu\nu}(\partial_\mu\partial_\nu\phi^i)\left(\Delta_{\widetilde{G}}\phi^j + \widetilde{\Gamma}^{\rho}(\partial_\rho\phi^j)\right)\delta_{ij} + 2G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}\widetilde{\Gamma}^{\eta}\widetilde{G}_{\lambda\eta} \\ &- G^{\mu\nu}G^{\rho\sigma}\Gamma^{\lambda}_{\mu\nu}\Gamma^{\eta}_{\rho\sigma}G_{\lambda\eta}\Big). \end{aligned} \tag{B.19}$$

General case: $\widetilde{G} \neq g$ but using e.o.m. We have

$$e^{\sigma}G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}\widetilde{\Gamma}^{\eta}G_{\lambda\eta} = G^{\mu\nu}G^{\rho\sigma}\Gamma^{\lambda}_{\mu\nu}\Gamma^{\lambda'}_{\rho\sigma}G_{\lambda\lambda'} = G^{\rho\sigma}g^{\delta\beta}\Gamma^{\eta}_{\delta\beta}\Gamma^{\lambda}_{\rho\sigma}g_{\eta\lambda} = 0. \tag{B.20}$$

This can be seen by considering on-shell configurations $\Delta_{\widetilde{G}}\phi^i = 0$, $\widetilde{\Gamma}^\mu = 0$ which imply

$$\begin{aligned} G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} &= G^{\mu\nu}(\partial_\rho\phi^i)(\partial_\mu\partial_\nu\phi^j)g^{\rho\lambda}\delta_{ij} \\ &= e^{\sigma}g^{\rho\lambda}(\partial_\rho\phi^i)\widetilde{G}^{\mu\nu}(\partial_\mu\partial_\nu\phi^j)\delta_{ij} \\ &= 0. \end{aligned} \tag{B.21}$$

We have shown that for on-shell geometries $\mathrm{tr}\mathcal{E}$ is indeed a covariant expression,

$$\begin{aligned} \mathrm{tr}\mathcal{E} &= -\frac{e^{-\sigma}}{4}\mathrm{tr}\, G_{\mu\nu}a^\mu a^\nu \\ &= -\frac{e^{-\sigma}}{4}\,k\,\Big(G^{\mu\nu}(\nabla_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\nabla_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\ &\quad + G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\Big) - \frac{k}{4}\widetilde{G}^{\mu\nu}R_{\mu\nu}[g]. \end{aligned} \tag{B.22}$$

Hence we can go to normal coordinates to simplify the comparison between tr$\mathcal{E}$ and the Ricci scalar $R[\widetilde{G}]$. Keep in mind that the covariant derivative in Eq. (B.22) is with respect to the background metric $g_{\mu\nu}$.

Special case: $\widetilde{G} = g$ without the use of e.o.m. In that case we have

$$G^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}\widetilde{\Gamma}^{\eta}\widetilde{G}_{\lambda\eta} = G^{\mu\nu}G^{\rho\sigma}\Gamma^{\lambda}_{\mu\nu}\Gamma^{\eta}_{\rho\sigma}G_{\lambda\eta} = g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}G^{\rho\sigma}\Gamma^{\eta}_{\rho\sigma}g_{\lambda\eta} = e^{\sigma}\Gamma^{\lambda}\Gamma^{\eta}g_{\lambda\eta}. \tag{B.23}$$

So we find for tr$\mathcal{E}$

$$\begin{aligned}
\mathrm{tr}\mathcal{E} &= -\frac{e^{-\sigma}}{4}\mathrm{tr}(a^{\mu}a^{\nu}G_{\mu\nu}) + \frac{k}{4}\Gamma^{\lambda}\Gamma^{\eta}g_{\lambda\eta} \\
&= -\frac{e^{\sigma}}{4}\,k\left(g^{\mu\nu}(\nabla_{\mu}\theta^{-1}_{\nu\alpha})g^{\rho\sigma}(\nabla_{\rho}\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - g^{\mu\nu}g^{\rho\sigma}(\nabla_{\mu}\theta^{-1}_{\rho\alpha})(\nabla_{\nu}\theta^{-1}_{\sigma\beta})g^{\alpha\beta}\right. \\
&\quad \left. + g^{\mu\nu}g^{\rho\sigma}(\nabla_{\mu}\theta^{-1}_{\rho\alpha})(\nabla_{\sigma}\theta^{-1}_{\nu\beta})g^{\alpha\beta}\right) - \frac{k}{4}g^{\mu\nu}R_{\mu\nu}[g] - \frac{1}{2}\Gamma^{\mu}\Gamma^{\nu}g_{\mu\nu} \\
&\quad + \frac{e^{-\sigma}}{4}k\left(\Delta_g\phi^i + \Gamma^{\mu}(\partial_{\mu}\phi^i)\right)\left(\Delta_g\phi^j + \Gamma^{\nu}(\partial_{\nu}\phi^j)\right)\delta_{ij} \\
&\quad + \frac{1}{4}k\,g_{\mu\nu}\Gamma^{\mu}\Gamma^{\nu}.
\end{aligned} \tag{B.24}$$

Noticing the following relation

$$\begin{aligned}
\Gamma^{\mu} &= g^{\mu\nu}g^{\rho\sigma}(\partial_{\rho}g_{\sigma\nu}) - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\partial_{\nu}g_{\rho\sigma} \\
&= g^{\mu\nu}g^{\rho\sigma}(\partial_{\rho}\partial_{\sigma}\phi^i)(\partial_{\nu}\phi^j)\delta_{ij} \\
&= g^{\mu\nu}(\partial_{\nu}\phi^j)\delta_{ij}\left(\Delta_g\phi^i + \Gamma^{\rho}(\partial_{\rho}\phi^i)\right)
\end{aligned} \tag{B.25}$$

or

$$g_{\rho\mu}\Gamma^{\mu} = (\partial_{\rho}\phi^i)\left(\Delta_g\phi^j + \Gamma^{\mu}(\partial_{\mu}\phi^j)\right)\delta_{ij}, \tag{B.26}$$

and recalling

$$\Delta_g x^a = \begin{pmatrix} \Delta_g x^{\mu} \\ \Delta_g\phi^i \end{pmatrix} = \begin{pmatrix} -\Gamma^{\mu} \\ \Delta_g\phi^i \end{pmatrix} \tag{B.27}$$

we see that

$$\partial_{\rho}x^a\Delta_g x^b\eta_{ab} = -\Gamma^{\mu}\eta_{\mu\rho} + \partial_{\rho}\phi^i\Delta_g\phi^j\delta_{ij}, \tag{B.28}$$

as well as

$$\begin{aligned}
g_{\rho\mu}\Gamma^{\mu} &= (\partial_{\rho}\phi^i)(\Delta_g\phi^j)\delta_{ij} + \Gamma^{\mu}(\partial_{\mu}\phi^i)(\partial_{\rho}\phi^j)\delta_{ij} \\
&= (\partial_{\rho}x^a)(\Delta_g x^b)\eta_{ab} + \Gamma_{\mu}\eta_{\mu\rho} + \Gamma^{\mu}(\partial_{\mu}\phi^i)(\partial_{\rho}\phi^j)\delta_{ij} \\
&= (\partial_{\rho}x^a)(\Delta_g x^b)\eta_{ab} + \Gamma^{\mu}g_{\mu\rho}.
\end{aligned} \tag{B.29}$$

Therefore we have

$$(\partial_\rho x^a)(\Delta_g x^b)\eta_{ab} = 0 \tag{B.30}$$

or

$$(\partial_\rho \phi^i)(\Delta_g \phi^j)\delta_{ij} = \Gamma^\mu \eta_{\mu\rho}. \tag{B.31}$$

It is worthwhile mentioning that the relation Eq. (B.30) for the general case $\widetilde{G} \neq g$ turns out to be an e.o.m [51] and thus this equation usually holds only for on-shell geometries.

$$\begin{aligned}
\left(\Delta_g\phi^i + \Gamma^\mu(\partial_\mu\phi^i)\right)\left(\Delta_g\phi^j + \Gamma^\nu(\partial_\nu\phi^j)\right)\delta_{ij} &= \Big((\Delta_g\phi^i)(\Delta_g\phi^j) + 2\Gamma^\mu(\partial_\mu\phi^i)(\Delta_g\phi^j) \\
&\quad + \Gamma^\mu\Gamma^\nu(\partial_\mu\phi^i)(\partial_\nu\phi^j)\Big)\delta_{ij} \\
&= (\Delta_g\phi^i)(\Delta_g\phi^j)\delta_{ij} + 2\Gamma^\nu\eta_{\mu\nu} \\
&\quad + \Gamma^\mu\Gamma^\nu(\partial_\mu\phi^i)(\partial_\nu\phi^j)\delta_{ij} \\
&= (\Delta_g\phi^i)(\Delta_g\phi^j)\delta_{ij} + \Gamma^\mu\Gamma^\nu\eta_{\mu\nu} \\
&\quad + \Gamma^\mu\Gamma^\nu g_{\mu\nu} \\
&= (\Delta_g\phi^i)(\Delta_g\phi^j)\delta_{ij} + (\Delta_g x^\mu)(\Delta_g x^\nu)\eta_{\mu\nu} \\
&\quad + \Gamma^\mu\Gamma^\nu g_{\mu\nu} \\
&= (\Delta_g x^a)(\Delta_g x^b)\eta_{ab} + \Gamma^\mu\Gamma^\nu g_{\mu\nu}
\end{aligned} \tag{B.32}$$

$$\begin{aligned}
\mathrm{tr}\mathcal{E} &= -\frac{e^{-\sigma}}{4}\mathrm{tr}(a^\mu a^\nu G_{\mu\nu}) + \frac{k}{4}\Gamma^\lambda\Gamma^\eta g_{\lambda\eta} \\
&= -\frac{e^\sigma}{4}\,k\,\Big(g^{\mu\nu}(\nabla_\mu\theta^{-1}_{\nu\alpha})g^{\rho\sigma}(\nabla_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&\quad + g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\Big) - \frac{k}{4}R[g] - \frac{1}{2}\Gamma^\mu\Gamma^\nu g_{\mu\nu} \\
&\quad + \frac{k}{4}(\Delta_g x^a)(\Delta_b x^b)\eta_{ab} + \frac{k}{4}\,\Gamma^\mu\Gamma^\nu g_{\mu\nu} + \frac{k}{4}\Gamma^\mu\Gamma^\nu g_{\mu\nu} \\
&= -\frac{e^\sigma}{4}\,k\,\Big(g^{\mu\nu}(\nabla_\mu\theta^{-1}_{\nu\alpha})g^{\rho\sigma}(\nabla_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
&\quad + g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\Big) - \frac{k}{4}R[g] + \frac{k}{4}(\Delta_g x^a)(\Delta_b x^b)\eta_{ab}.
\end{aligned} \tag{B.33}$$

Due to the antisymmetry of $\theta^{\mu\nu}$ and since $\theta^{\mu\nu}$ also fulfills the Jacobi identity we have

$$\nabla_\rho\theta^{-1}_{\mu\nu} + \nabla_\nu\theta^{-1}_{\rho\mu} + \nabla_\mu\theta^{-1}_{\nu\rho} = 0 \tag{B.34}$$

Via the following computation

$$\begin{aligned}G^{\mu\nu}G^{\rho\sigma}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} &= G^{\mu\nu}G^{\rho\sigma}\left(\nabla_\alpha\theta^{-1}_{\mu\rho}+\nabla_\rho\theta^{-1}_{\alpha\mu}\right)\left(\nabla_\beta\theta^{-1}_{\nu\sigma}+\nabla_\sigma\theta^{-1}_{\beta\nu}\right)g^{\alpha\beta}\\ &= G^{\mu\nu}G^{\rho\sigma}\Big((\nabla_\alpha\theta^{-1}_{\mu\rho})(\nabla_\beta\theta^{-1}_{\nu\sigma})+2(\nabla_\rho\theta^{-1}_{\alpha\mu})(\nabla_\beta\theta^{-1}_{\nu\sigma})\\ &\quad+(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})\Big)g^{\alpha\beta}\end{aligned} \tag{B.35}$$

we see that

$$G^{\mu\nu}G^{\rho\sigma}\Big((\nabla_\alpha\theta^{-1}_{\mu\rho})(\nabla_\beta\theta^{-1}_{\nu\sigma})+2(\nabla_\rho\theta^{-1}_{\alpha\mu})(\nabla_\beta\theta^{-1}_{\nu\sigma})\Big)g^{\alpha\beta}=0 \tag{B.36}$$

In the case of $\widetilde{G}_{\mu\nu}=g_{\mu\nu}$ we hence have

$$g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\nu\theta^{-1}_{\sigma\beta})=2g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}(\nabla_\mu\theta^{-1}_{\rho\alpha})(\nabla_\sigma\theta^{-1}_{\nu\beta}). \tag{B.37}$$

Also, in that case the following relation holds

$$g^{\mu\nu}\nabla_\mu\theta^{-1}_{\nu\rho}=0. \tag{B.38}$$

To see this consider the covariant derivative acting on the Poisson structure

$$g^{\mu\nu}\nabla_\mu\theta^{-1}_{\nu\rho}=g^{\mu\nu}\partial_\mu\theta^{-1}_{\nu\rho}-g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\theta^{-1}_{\lambda\rho}-g^{\mu\nu}\Gamma^\lambda_{\mu\rho}\theta^{-1}_{\nu\lambda}. \tag{B.39}$$

Using

$$\begin{aligned}g^{\mu\nu}\Gamma^\lambda_{\mu\rho}\theta^{-1}_{\lambda\nu} &= \Big(\frac{1}{2}g^{\mu\nu}g^{\lambda\eta}(\partial_\mu g_{\eta\rho})-\frac{1}{2}g^{\mu\nu}g^{\lambda\eta}(\partial_\eta g_{\mu\rho})-\frac{1}{2}(\partial_\rho g^{\lambda\nu})\Big)\theta^{-1}_{\nu\lambda}\\ &= g^{\mu\nu}g^{\lambda\eta}(\partial_\mu g_{\eta\rho})\theta^{-1}_{\nu\lambda}\\ &= -e^{-\sigma}\theta^{\mu\eta}(\partial_\mu g_{\eta\rho})\end{aligned} \tag{B.40}$$

we can see that

$$\begin{aligned}g^{\mu\nu}\nabla_\mu\theta^{-1}_{\nu\rho} &= g^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\rho})+e^{-\sigma}\theta^{\mu\eta}(\partial_\mu g_{\nu\rho})-\Gamma^\lambda\theta^{-1}_{\lambda\rho}\\ &= g^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\rho})+e^{-\sigma}\theta^{\mu\eta}(\partial_\mu g_{\nu\rho})\\ &\quad+e^{-\sigma}(\partial_\eta\theta^{\lambda\alpha})\theta^{\eta\beta}g_{\alpha\beta}\theta^{-1}_{\lambda\rho}+e^{-\sigma}\theta^{\lambda\alpha}\theta^{\eta\beta}(\partial_\rho g_{\alpha\beta})\theta^{-1}_{\lambda\rho}\\ &= 0\end{aligned} \tag{B.41}$$

The covariant e.o.m. that was derived in [50]

$$\widetilde{G}^{\mu\nu}\widetilde{\nabla}_\mu(e^\sigma\theta^{-1}_{\nu\rho})=\frac{e^{-\sigma}}{4}\widetilde{G}_{\rho\mu}\theta^{\mu\nu}\partial_\nu(G^{\kappa\lambda}g_{\kappa\lambda}) \tag{B.42}$$

reduces for $\widetilde{G}_{\mu\nu}=g_{\mu\nu}$ to

$$g^{\mu\nu}\nabla_\mu\theta^{-1}_{\nu\rho}=0, \tag{B.43}$$

which has the form of a homogeneous Maxwell equation. So for $\widetilde{G} = g$ this relation is actually an identity.

Our final result for tr$\mathcal{E}$ in case of $\widetilde{G}_{\mu\nu} = g_{\mu\nu}$ is

$$\mathrm{tr}\mathcal{E} = \frac{e^\sigma}{4} k\, g^{\mu\nu} g^{\rho\sigma} (\nabla_\mu \theta^{-1}_{\rho\alpha})(\nabla_\sigma \theta^{-1}_{\nu\beta}) g^{\alpha\beta} - \frac{k}{4} R[g] + \frac{k}{4} (\Delta_g x^a)(\Delta_g x^b)\eta_{ab}. \tag{B.44}$$

Thus we have shown that in the case of $\widetilde{G}_{\mu\nu} = g_{\mu\nu}$, for a covariance proof it is not necessary to use e.o.m.

Special case $\widetilde{G} = g$ using e.o.m. tr$\mathcal{E}$ is now very simple,

$$\mathrm{tr}\mathcal{E} = \frac{e^\sigma}{4} k\, g^{\mu\nu} g^{\rho\sigma} (\nabla_\mu \theta^{-1}_{\rho\alpha})(\nabla_\sigma \theta^{-1}_{\nu\beta}) g^{\alpha\beta} - \frac{k}{4} R[g]. \tag{B.45}$$

Appendix C

R in normal coordinates

We evaluate the Ricci scalar in normal coordinates. First of all note that

$$\widetilde{G}^{\rho\sigma}\partial_\mu\widetilde{G}_{\rho\sigma} = g^{\rho\sigma}\partial_\mu g_{\rho\sigma} \overset{\text{nc}}{=} 0. \tag{C.1}$$

Using also

$$\widetilde{G}^{\mu\nu}\widetilde{G}^{\rho\sigma}(\partial_\mu\partial_\rho\widetilde{G}_{\nu\sigma}) = -\widetilde{G}^{\mu\nu}(\partial_\mu\widetilde{G}^{\rho\sigma})(\partial_\rho\widetilde{G}_{\sigma\nu}) + \widetilde{G}_{\mu\nu}(\partial_\rho\widetilde{G}^{\rho\mu})(\partial_\sigma\widetilde{G}^{\sigma\nu}) - \partial_\mu\partial_\nu\widetilde{G}^{\mu\nu} \tag{C.2}$$

we can simplify the Ricci scalar Eq.(4.56) and we obtain

$$\begin{aligned} R[\widetilde{G}] = e^{-\sigma}\Big\{ &-\frac{3}{2}G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) - 3G^{\mu\nu}(\partial_\mu\partial_\nu\sigma) \\ &+ (\partial_\mu G^{\mu\nu})(\partial_\nu\sigma) - \frac{1}{2}G^{\mu\nu}(\partial_\mu\sigma)(G^{\rho\sigma}\partial_\nu G_{\rho\sigma}) \\ &- \frac{1}{2}G^{\mu\nu}(\partial_\mu G^{\rho\sigma})(\partial_\rho G_{\sigma\nu}) - \partial_\mu\partial_\nu G^{\mu\nu} \\ &- G^{\mu\nu}(G^{\rho\sigma}\partial_\mu\partial_\nu G_{\rho\sigma}) - \frac{3}{4}G^{\mu\nu}(\partial_\mu G^{\rho\sigma})(\partial_\nu G_{\rho\sigma})\Big\}. \end{aligned} \tag{C.3}$$

Next give a list of terms that appear in the Ricci scalar in normal coordinates. Also here we exploit the e.o.m. Eq. (3.52).

$$\begin{aligned} G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) &= (\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} \\ G^{\mu\nu}(\partial_\mu\partial_\nu\sigma) &= \frac{1}{2}G^{\mu\nu}(\theta^{\rho\sigma}\partial_\mu\partial_\nu\theta_{\rho\sigma}) + \frac{1}{2}G^{\mu\nu}(\partial_\mu\theta^{\rho\sigma})(\partial_\nu\theta_{\rho\sigma}) \\ &\quad + \frac{1}{2}G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma}) \\ (\partial_\mu G^{\mu\nu})(\partial_\nu\sigma) &\overset{\text{e.o.m.}}{=} (\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} \\ (G^{\rho\sigma}\partial_\mu G_{\rho\sigma}) &= -4(\partial_\mu\sigma) \\ G^{\mu\nu}(\partial_\mu\sigma)(G^{\rho\sigma}\partial_\nu G_{\rho\sigma}) &= -4(\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} \end{aligned} \tag{C.4}$$

$$
\begin{aligned}
G^{\mu\nu}(\partial_\mu G^{\rho\sigma})(\partial_\nu G_{\rho\sigma}) &= -2G^{\mu\nu}(\partial_\mu\theta^{\rho\sigma})(\partial_\nu\theta_{\rho\sigma}) - 2G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta}\\
(\partial_\mu G^{\rho\sigma})(\partial_\nu G_{\rho\sigma}) + G^{\rho\sigma}\partial_\mu\partial_\nu G_{\rho\sigma} &= -4\partial_\mu\partial_\nu\sigma + g^{\rho\sigma}(\partial_\mu\partial_\nu g_{\rho\sigma})\\
G^{\mu\nu}(G^{\rho\sigma}\partial_\mu\partial_\nu G_{\rho\sigma}) &= -G^{\mu\nu}(\partial_\mu G^{\rho\sigma})(\partial_\nu G_{\rho\sigma}) - 4\partial_\mu\partial_\nu\sigma + G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma})\\
&= 2G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - 2G^{\mu\nu}(\theta^{\rho\sigma}\partial_\mu\partial_\nu\theta_{\rho\sigma})\\
&\quad - G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma})
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
G^{\mu\nu}(\partial_\mu G^{\rho\sigma})(\partial_\rho G_{\sigma\nu}) &= -(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} - 2G^{\mu\nu}(\partial_\mu\theta^{\rho\alpha})(\partial_\rho\theta^{-1}_{\nu\alpha})\\
&\quad - G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta}\\
\partial_\mu\partial_\nu G^{\mu\nu} &\overset{\text{e.o.m.}}{=} \theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + (\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta}\\
&= \frac{1}{2}G^{\mu\nu}(\theta^{\rho\sigma}\partial_\mu\partial_\nu\theta_{\rho\sigma}) + \frac{1}{2}G^{\mu\nu}(\partial_\mu\theta^{\rho\sigma})(\partial_\nu\theta_{\rho\sigma})\\
&\quad + (\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta}
\end{aligned}
\tag{C.6}
$$

Using these we finally obtain

$$
\begin{aligned}
R[\widetilde{G}] \overset{\text{nc}}{=} e^{-\sigma}\Big\{&\frac{1}{2}(\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + \frac{1}{2}(\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta}\\
&+ \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} - \frac{1}{2}G^{\mu\nu}G^{\rho\sigma}(\partial_\mu\theta^{-1}_{\rho\alpha})(\partial_\nu\theta^{-1}_{\sigma\beta})g^{\alpha\beta}\\
&- \frac{1}{2}G^{\mu\nu}(g^{\rho\sigma}\partial_\mu\partial_\nu g_{\rho\sigma})\Big\}.
\end{aligned}
\tag{C.7}
$$

Appendix D

Evaluation of tr$\mathcal{E}$ and $R[\widetilde{G}]$ for $D = 4$

We quote some identities which appear in the computation of $R[\widetilde{G}]$ in terms of θ-vielbeins

$$\begin{aligned}
(\partial_\beta G^{\beta\mu})G^{\nu\alpha}(\partial_\alpha G_{\mu\nu}) &= -G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} - 2(\partial_\alpha\theta^{\alpha\rho})G^{\beta\gamma}(\partial_\beta\theta^{-1}_{\gamma\rho}) \\
&\quad - (\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} \\
G^{\alpha\beta}G^{\mu\nu}\partial_\beta\partial_\nu G_{\alpha\mu} &= 2G^{\mu\rho}G^{\nu\sigma}\theta^{-1}_{\mu\alpha}(\partial_\rho\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} + G^{\beta\gamma}(\partial_\gamma\theta^{-1}_{\alpha\rho})G^{\alpha\lambda}(\partial_\lambda\theta^{-1}_{\beta\sigma})g^{\rho\sigma} \\
&\quad G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\alpha})G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
G^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma G_{\mu\nu} &= -2\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} + 2G^{\mu\nu}G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\mu\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
(\partial_\mu G^{\mu\nu})(\partial_\nu\sigma) &= G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) + \theta^{\beta\gamma}(\partial_\beta\theta^{\lambda\alpha})(\partial_\lambda\sigma)g_{\alpha\gamma} \\
G^{\rho\sigma}(\partial_\rho\theta^{\mu\nu})(\partial_\sigma\theta^{-1}_{\mu\nu}) &= -2G^{\rho\sigma}(\partial_\rho\theta^{\mu\nu})(\partial_\sigma\theta^{-1}_{\mu\nu}) - 2G^{\mu\nu}G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\mu\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
G^{\nu\rho}(\partial_\rho G^{\alpha\gamma})(\partial_\gamma G_{\nu\alpha}) &= -2G^{\mu\rho}(\partial_\rho\theta^{\nu\sigma})(\partial_\nu\theta^{-1}_{\mu\sigma}) - (\partial_\nu\theta^{\gamma\sigma})(\partial_\gamma\theta^{\nu\rho})g_{\rho\sigma} \\
&\quad - G^{\nu\kappa}(\partial_\kappa\theta^{-1}_{\gamma\rho})G^{\gamma\lambda}(\partial_\lambda\theta^{-1}_{\nu\sigma})g^{\rho\sigma} \\
G^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) &= \frac{1}{4}G^{\rho\sigma}(\theta^{\mu\nu}\partial_\rho\theta^{-1}_{\mu\nu})(\theta^{\kappa\lambda}\partial_\sigma\theta^{-1}_{\kappa\lambda}) \\
G^{\mu\nu}\partial_\mu\partial_\nu\sigma &= \frac{1}{2}G^{\rho\sigma}(\partial_\rho\theta^{\mu\nu})(\partial_\sigma\theta^{-1}_{\mu\nu}) + \frac{1}{2}\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu}
\end{aligned} \tag{D.1}$$

Below we give some identities which do not appear in the computation of the Ricci scalar but are important for tr$\mathcal{E}$:

$$\begin{aligned}
G^{\mu\nu}(\partial_\mu G_{\nu\rho})\theta^{\rho\alpha}(\partial_\rho\theta^{\sigma\beta})g_{\alpha\beta} &= -G^{\mu\kappa}(\partial_\kappa\theta^{-1}_{\mu\alpha})G^{\nu\lambda}(\partial_\lambda\theta^{-1}_{\nu\beta})g^{\alpha\beta} - (\partial_\mu\theta^{\mu\alpha})G^{\nu\lambda}(\partial_\nu\theta^{-1}_{\lambda\alpha}) \\
\partial_\mu\partial_\nu G^{\mu\nu} &= 2\,\theta^{\mu\alpha}(\partial_\mu\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} + (\partial_\mu\theta^{\nu\alpha})(\partial_\nu\theta^{\mu\beta})g_{\alpha\beta} \\
&\quad + (\partial_\mu\theta^{\mu\alpha})(\partial_\nu\theta^{\nu\beta})g_{\alpha\beta} \\
G_{\nu\kappa}\theta^{\sigma\alpha}(\partial_\sigma\theta^{\nu\beta})\theta^{\lambda\gamma}(\partial_\lambda\theta^{\kappa\delta})g_{\alpha\beta}g_{\gamma\delta} &= G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha})G^{\rho\sigma}(\partial_\rho\theta^{-1}_{\sigma\beta})g^{\alpha\beta} \\
G_{\nu\kappa}\theta^{\sigma\alpha}(\partial_\sigma\theta^{\nu\beta})\theta^{\lambda\gamma}(\partial_\lambda\theta^{\kappa\delta})g_{\alpha\gamma}g_{\beta\delta} &= G^{\rho\sigma}G^{\mu\nu}(\partial_\rho\theta^{-1}_{\mu\alpha})(\partial_\sigma\theta^{-1}_{\nu\beta})g^{\alpha\beta} \\
G_{\nu\kappa}\theta^{\sigma\alpha}(\partial_\sigma\theta^{\nu\beta})\theta^{\lambda\gamma}(\partial_\lambda\theta^{\kappa\delta})g_{\alpha\delta}g_{\beta\gamma} &= G^{\mu\rho}(\partial_\rho\theta^{-1}_{\nu\alpha})G^{\nu\sigma}(\partial_\sigma\theta^{-1}_{\mu\beta})g^{\alpha\beta} \\
G_{\mu\nu}(\partial_\rho G^{\rho\mu})(\partial_\sigma G^{\sigma\nu}) &= (\partial_\rho\theta^{\rho\alpha})(\partial_\sigma\theta^{\sigma\beta})g_{\alpha\beta} + 2\,(\partial_\rho\theta^{\rho\alpha})G^{\mu\nu}(\partial_\mu\theta^{-1}_{\nu\alpha}) \\
&\quad + G^{\mu\kappa}(\partial_\kappa\theta_{\mu\alpha})G^{\nu\lambda}(\partial_\lambda\theta_{\nu\beta})g^{\alpha\beta}
\end{aligned} \tag{D.2}$$

Appendix E

Expressing R and tr$\mathcal{E}$ in $\bar{x}$-coordinates

We rewrite the terms which compose the Ricci scalar R and tr$\mathcal{E}$ in terms of the $U(1)$ gauge fields. We need [48]

$$
\begin{aligned}
e^{\sigma} &= (\det G^{\mu\nu})^{1/4} = (\det \bar{g}^{\mu\nu})^{1/4}\left(1 - \frac{1}{2}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu} + O(\bar{\theta}^2)\right)\\
\theta^{\mu\nu} &= \bar{\theta}^{\mu\nu} - \bar{\theta}^{\mu\alpha}\bar{\theta}^{\nu\beta}\bar{F}_{\alpha\beta}\\
\theta^{-1}_{\mu\nu} &= \bar{\theta}^{-1}_{\mu\nu} - \bar{F}_{\mu\nu}\\
G^{\mu\nu} &= (\bar{\theta}^{\alpha\gamma} - \bar{\theta}^{\alpha\mu}\bar{\theta}^{\gamma\nu}\bar{F}_{\mu\nu})(\bar{\theta}^{\beta\delta} - \bar{\theta}^{\beta\rho}\bar{\theta}^{\delta\sigma}\bar{F}_{\rho\sigma})g_{\gamma\delta}
\end{aligned}
\tag{E.1}
$$

Below we give a list of terms in terms of $\bar{\theta}^{\mu\nu}$ in $\bar{x}$-coordinates to $O(\bar{A}^2)$. We denote $|\det \bar{g}_{\alpha\beta}| \equiv |\bar{g}|$.

$$
\begin{aligned}
\int \mathrm{d}^4x\, e^{-\sigma}\, G^{\mu\kappa}\left(\partial_\kappa\theta^{-1}_{\mu\alpha}\right)G^{\nu\lambda}\left(\partial_\lambda\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta} &= -\frac{1}{4}\int \mathrm{d}^4\bar{x}\, |\bar{g}|^{-1/4}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^{\alpha}\bar{\partial}_{\alpha}\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\\
\int \mathrm{d}^4x\, e^{-\sigma}\,\left(\partial_\mu\theta^{\mu\alpha}\right)G^{\nu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right) &= -\frac{1}{4}\int \mathrm{d}^4\bar{x}\, |\bar{g}|^{-1/4}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^{\alpha}\bar{\partial}_{\alpha}\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\\
\int \mathrm{d}^4x\, e^{-\sigma}\,\left(\partial_\mu\theta^{\nu\alpha}\right)\left(\partial_\nu\theta^{\mu\beta}\right)g_{\alpha\beta} &= -\frac{1}{4}\int \mathrm{d}^4\bar{x}\, |\bar{g}|^{-1/4}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^{\alpha}\bar{\partial}_{\alpha}\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\\
\int \mathrm{d}^4x\, e^{-\sigma}\,\theta^{\mu\alpha}\partial_\mu\partial_\nu\theta^{\nu\beta}g_{\alpha\beta} &= \int \mathrm{d}^4\bar{x}\; -|\bar{g}|^{-1/4}\Big(\frac{1}{2}\bar{\partial}^{\alpha}\bar{\partial}_{\alpha}\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\\
&\quad + \frac{1}{4}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^{\alpha}\bar{\partial}_{\alpha}\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\Big)
\end{aligned}
\tag{E.2}
$$

$$
\begin{aligned}
\int \mathrm{d}^4x\, e^{-\sigma}\, G^{\rho\sigma}\left(\partial_\rho\theta^{\mu\nu}\right)\left(\partial_\sigma\theta^{-1}_{\mu\nu}\right) &= -\frac{1}{2}\int \mathrm{d}^4\bar{x}\, |\bar{g}|^{-1/4}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^\alpha\bar{\partial}_\alpha\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\\
\int \mathrm{d}^4x\, e^{-\sigma}\,\theta^{\mu\nu}G^{\rho\sigma}\partial_\rho\partial_\sigma\theta^{-1}_{\mu\nu} &= \int \mathrm{d}^4\bar{x}\, |\bar{g}|^{-1/4}\Big(-\bar{\partial}^\alpha\bar{\partial}_\alpha\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\\
&\quad + \frac{1}{2}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^\alpha\bar{\partial}_\alpha\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\Big)\\
\int \mathrm{d}^4x\, e^{-\sigma}\, G^{\kappa\lambda}G^{\mu\nu}\left(\partial_\kappa\theta^{-1}_{\mu\alpha}\right)\left(\partial_\lambda\theta^{-1}_{\nu\beta}\right)g^{\alpha\beta} &= \int \mathrm{d}^4\bar{x}\, |\bar{g}|^{-1/4}\Big(-\frac{1}{4}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^\alpha\bar{\partial}_\alpha\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}\\
&\quad - \frac{1}{2}g^{\alpha\gamma}g^{\beta\delta}\bar{F}_{\alpha\beta}\bar{\partial}^2\bar{F}_{\gamma\delta}\Big)\\
\int \mathrm{d}^4x\, e^{-\sigma}\, G^{\mu\kappa}\left(\partial_\kappa\theta^{-1}_{\nu\alpha}\right)G^{\nu\lambda}\left(\partial_\lambda\theta^{-1}_{\mu\beta}\right)g^{\alpha\beta} &= -\frac{1}{4}\int \mathrm{d}^4\bar{x}\, |\bar{g}|^{-1/4}\bar{\theta}^{\mu\nu}\bar{F}_{\mu\nu}\bar{\partial}^\alpha\bar{\partial}_\alpha\bar{\theta}^{\rho\sigma}\bar{F}_{\rho\sigma}
\end{aligned}
\tag{E.3}
$$

where we used [48]

$$
\begin{aligned}
\int \mathrm{d}^4\bar{x}\,\bar{F}_{\mu\nu}\bar{\theta}^{\mu\nu}\bar{\partial}^\alpha\bar{\partial}_\alpha\bar{F}_{\rho\sigma}\bar{\theta}^{\rho\sigma} &= \int \mathrm{d}^4\bar{x}\, g^{\mu\nu}\bar{F}_{\mu\alpha}\bar{\partial}^\alpha\bar{\partial}^\rho\bar{F}_{\nu\rho},\\
\int \mathrm{d}^4\bar{x}\, 4\, g^{\alpha\beta}\bar{F}_{\alpha\mu}\bar{\partial}^\mu\bar{\partial}^\nu\bar{F}_{\beta\nu} &= \int \mathrm{d}^4\bar{x}\,\left(-g^{\sigma\mu}\bar{g}^{\alpha\nu}\bar{F}_{\nu\sigma}\bar{\partial}^\lambda\bar{\partial}_\lambda\bar{F}_{\mu\alpha} + \frac{1}{2}\bar{g}^{\alpha\nu}\bar{g}^{\mu\rho}\bar{F}_{\nu\mu}\bar{\partial}^2\bar{F}_{\rho\alpha}\right).
\end{aligned}
\tag{E.4}
$$

Appendix F

Danksagung

Die letzten Tastenschläge klingen an. Zeit, vielen Menschen *danke* zu sagen. Ganz besonders Harald & Harold.

Meinem Doktorvater Harald Grosse danke ich dafür, dass er mir in vielen langen Gesprächen gezeigt hat, was es alles noch zu verstehen gibt und dafür, dass er mit seiner Begeisterung das Herz der nichtkommutativen Gruppe ist. Dafür, dass ich in dieser Gruppe lernen durfte und dafür, dass er mit unermüdlichem Einsatz diese Gruppe erhält.

Meinem zweiten Doktorvater Harold Steinacker danke ich dafür, dass er mir gezeigt hat, wie wichtig Intuition als treibende Kraft neuer physikalischer Welten ist. Dafür, dass er jede Frage beantworten konnte. Danke, Harold, aber vor allem für unsere fruchtbare Zusammenarbeit. Es war so schön.

Ein großes Dankeschön gebührt auch meinen angenehmen Schreibtischnachbarn Paul, Thomas und zuletzt Georg. Paul danke ich für die vielen Fragen, auf die ich zwar selten eine Antwort hatte, die aber immer interessante Diskussionen herbeiführten.

Bei den Mitgliedern der Arbeitsgruppe für Mathematische Physik bedanke ich mich für den regen Austausch am Mittagstisch.

Den Mitarbeitern des fünften Stocks, insbesondere Frau Picek, Frau Bogoevski und Herrn Moser möchte ich für sämtliche Unterstützung danken und dafür, dass sie allesamt diesem Stock seinen Charme verleihen.

Ich möchte mich weiters bei der Geschäftsstelle des Österreichischen Wissenschaftsrates für ihr Verständnis bedanken, dass diese Arbeit in den letzten Monaten noch in der Entstehung war.

Meiner Familie danke ich für andauernde, liebevolle Unterstützung.

Dem Glück danke ich, dass es an meiner Seite ist.

Diese Arbeit wurde durch das FWF-Projekt P20017 gefördert.

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Talks & Posters

June 2009	Poster: “Cosmology in Emergent Gravity”, Vienna Symposium on Fundamental Physics
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February 2009	Talk: “QCD-Instantons and Inversion Symmetry”, Schladming Winter School on Particle Physics
August 2008	Talk: “Fermions coupled to Noncommutative Emergent Gravity”,Workshop “Black Holes in General Relativity and String Theory”, Losinj, Croatia
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December 2007	Poster: “Generalized Feynman Rules and Unitarity”, Vienna Central European Seminar on Particle Physics and Quantum Field Theory
December 2006	Talk: “QCD-Instantons”, Technical University of Vienna
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Popular science publications

December 2008	"Julius Wess – a Life Concerened with Symmetry", *bridges* Vol. 20, http://www.ostina.org/content/view/3759/1111/
	"Introducing Astrophysicist Lisa Kaltenegger – Simulating Worlds Stranger Than Fiction", *bridges* Vol. 20, http://www.ostina.org/content/view/3756/1118/

Scholarships and awards

2008	Alfred-Wehrl-Award for Mathematical Physics
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